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MERCURY MERCAPTIDE NITRITES AND THEIR REACTION WITH THE ALKYL IODIDES.

Part IV

CHAIN COMPOUNDS OF SULPHUR—*continued*.

BY

PRAFULLA CHANDRA RÂY AND PRAFULLA CHANDRA GUHA.

In the previous communication of the present series (Pt. III. Trans. Chem. Soc. 1917, *111*, 101) it has been shown that whenever a potential mercaptan (*e.g.* thioacetamide, thiobenzamide, thiourea *etc.*), is treated with mercuric nitrite, the radical ($-SHgNO_2$) as a rule, gets detached and assumes the stable form $[3(SHgNO_2) HgO]_2$, whilst the organic component is converted into an aldehyde as also the corresponding acid with elimination of ammonia or substituted ammonia as the case may be.

In the present investigation the reactions of a large number of actual and potential mercaptans as also of dimercaptans, some of them belonging to the cyclo-group, have been studied. It was expected that in these the molecules being of a more complex nature, the radical specially ($-SHgNO_2$), would far more readily part company with the parent substance and lead an independent existence in the shape of $[3(SHgNO_2)HgO]$. The result has proved to be just the reverse. Mercurimercaptide nitrites of the ali-and heterocyclic compounds have been found to be stable compounds. A few typical cases will make the point clear. Phenyldithiobiazoline sulphydrate,

$$\begin{array}{c} PhN \cdot N \\ \quad \quad \| \\ \mid \quad CSH \\ \quad \quad \mid \\ SC - S \end{array}$$

yields with mercuric nitrite the corresponding mercaptide nitrite, the reaction being of the ordinary type;

$$RS\overline{H} + \overline{NO_2}Hg\ NO_2 = RSHgNO_2 + HNO_2 \text{ (Cf. Trans. Chem. Soc.}$$

1916, *109*, 131).
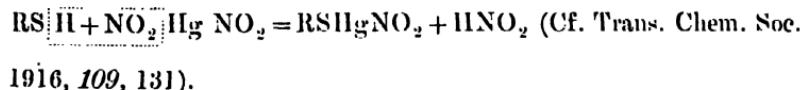
When a dimercaptan like thiobiazole-disulphydrate is similarly treated the reaction follows indentical lines but owing to the ready oxidisability of this class of compounds, it advances a step further, giving rise to a remarkable series of compounds containing sometimes as many as six or even eight sulphur atoms in a chain, thus :—

$$
\begin{array}{cccc}
N-N & N-N & N-N & N-N \\
\overset{\|}{\text{C}}\ \overset{\|}{\text{C}}S & S\overset{\|}{\text{C}}\ \overset{\|}{\text{C}}S & S\overset{\|}{\text{C}}\ \overset{\|}{\text{C}}S & S\overset{\|}{\text{C}}\ \overset{\|}{\text{C}}S \\
\text{NO}_2\text{Hg·C}\quad \overset{H\ H}{\underset{O}{}}\quad \overset{H\ H}{\underset{O}{}}\quad \overset{H\ H}{\underset{O}{}}\quad \text{HgNO}_2 \\
\text{V} & \text{V} & \text{V} & \text{V} \\
\text{S} & \text{S} & \text{S} & \text{S}
\end{array}
$$
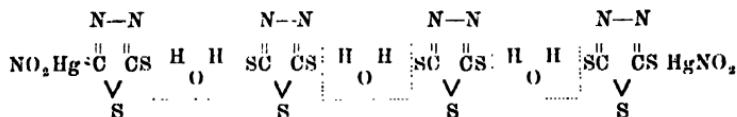
What happens is that often four molecules simultaneously take part in the reaction, the extreme members are converted into mercaptide nitrites and the nitrous acid set free reacts in turn upon the remaining (SH) groups, reproducing the following compound, which, it will be readily seen, is a tetradisulphide-dimercaptide-dinitrite :—

$$
\begin{array}{cccc}
N-N & N-N & N-N & N-N \\
S-C\ \ C-S-S-C\ \ C\ \ S-S-C\ \ C\ \ S-S\ \ C\ \ C-S \\
|\quad \text{V} & \text{V} & \text{V} & \text{V} \\
|\quad \text{S} & \text{S} & \text{S} & \text{S} \\
\text{Hg—O—NO} & & & \text{O}_2\text{N——Hg}
\end{array}
$$

$$
\begin{array}{cccc}
N-N & N-N & N-N & N-N \\
S-C\ \ C-S-S-\ C\ \ C-S-S-C\ \ C-S-S-C\ \ C-S \\
|\quad \text{V} & \text{V} & \text{V} & \text{V} \\
|\quad \text{S} & \text{S} & \text{S} & \text{S} \\
\text{Hg————O————Hg}
\end{array}
$$
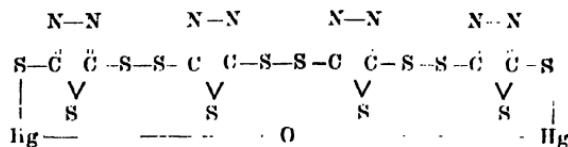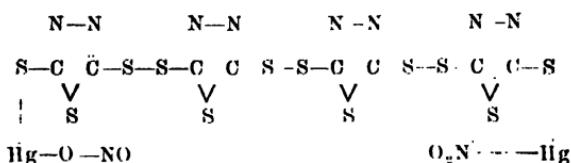
But as the molecule is a very heavy one it cannot retain the load of two (NO_2)'s; N_2O_3 (*i.e.*, $\text{NO}_2 + \text{NO}$) is evolved and an oxy compound is ultimately formed (Cf. Trans. Chem. Soc. 1916, *109*, 133).

Sometimes, the oxidation does not go so far and is limited to only three molecules resulting in the formation of a tridisulphide-dimercaptide-dinitrite or rather its oxysalt as shown above,

$$
\begin{array}{ccc}
N-N & N-N & N-N \\
\overset{\|}{\text{C}}\ \overset{\|}{\text{C}} & \overset{\|}{\text{C}}\ \overset{\|}{\text{C}} & \overset{\|}{\text{C}}\ \overset{\|}{\text{C}} \\
S-C\ \ C-S-S-C\ \ C-S-S-C\ \ C-S \\
|\quad \text{V} & \text{V} & \text{V} & | \\
|\quad \text{S} & \text{S} & \text{S} & | \\
\text{Hg——————O——————Hg}
\end{array}
$$
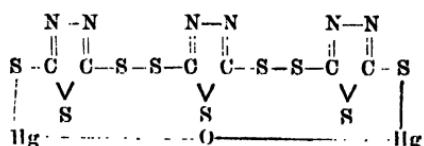
it was reasonably expected that if instead of the dimercaptan itself, its potassium salt were substituted the corresponding nitrite-mercaptide would be formed, as, in this case, in place of nitrous acid

its potassium salt would be generated and thus the chances of condensation of two or more molecules would be eliminated. This expectation has been realised. A compound of the formula

$$\text{KS} \cdot \overset{\overset{\displaystyle N{-}N}{\underset{}{\| \quad \|}}}{\underset{\underset{\displaystyle S}{V}}{C} \quad C} \cdot \text{SHgNO}_2$$
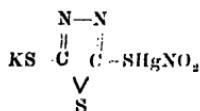
has been isolated. As the potassium-dimercaptide in aqueous solution undergoes partial hydrolysis reproducing the original mercaptan itself, it has also been found that when on keeping it for some-time it is acted upon by mercuric nitrite a compound of the formula

$$S{-}\overset{\overset{\displaystyle N{-}N}{\| \quad \|}}{\underset{\underset{S}{V}}{C} \quad C} \quad S{-}S{-}\overset{\overset{\displaystyle N{-}N}{\| \quad \|}}{\underset{\underset{S}{V}}{C} \quad C} \quad S$$
$$\text{Hg} \cdots\cdots\text{O} \qquad \cdots\text{Hg}$$
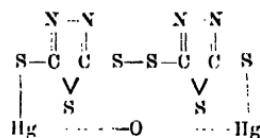
is formed. In this instance, as the nitrous acid available for oxidation is necessarily limited in quantity, a condensation of only two molecules takes place. It is noteworthy that when one molecule takes part in the reaction as in the previous case a real nitrite-mercaptide is formed, but only one-$HgNO_2$ group is fixed.

Whenever, however, two, three, or four molecules coalesce into a complex one the latter acquires the capacity to attach to the extreme ends of the chain two $-HgNO_2$ radicals ; but as the heavily laden molecule has to part with a molecule of N_2O_3, an oxy-mercuri-salt is the ultimate product. (Cf. Trans. Chem. Soc. 1917, *111*, 104).

The view taken above receives additional support from the fact that even aërial oxidation suffices to bring about condensation of three molecules ; for instance, when the above di-mercaptan is treated with mercuric chloride, a mercaptide chloride of the formula

$$\text{Cl Hg}{-}\text{S}{-}\overset{\overset{N{-}N}{\| \ \|}}{\underset{\underset{S}{V}}{C} \ C}{-}\text{S}{-}\text{S}{-}\overset{\overset{N{-}N}{\| \ \|}}{\underset{\underset{S}{V}}{C} \ C}{-}\text{S} \quad \text{S}{-}\overset{\overset{N{-}N}{\| \ \|}}{\underset{\underset{S}{V}}{C} \ C}{-}\text{S}{-}\text{HgCl}$$
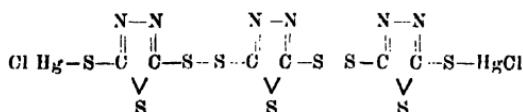
is formed and has been, in fact, actually isolated. It is thus evident that nitrous acid is by no means essential for the formation of the above chain compounds.

No less interesting is the behaviour of mercuric-nitrite towards thiophenol, which, as is well known, readily undergoes aërial oxidation and passes on to the stable disulphide form, $Ph_2 S_2$. Besides the expected $PhSHgNO_2$, a remarkable compound, Ph_2S_3Hg, is obtained and sometimes another oxysalt corresponding to the empirical formula, $[3(PhS), HgO]_2$. It is only in exceptional cases (vide "Experimental" p. 13) that the expected nitrite mercaptide, $PhSHgNO_2$ is obtained but only in an impure form.

The reaction would seem to take place in the following stages :—

$$PhSH + NO_2HgNO_2 = PhSHgNO_2 + HNO_2 \quad . . \quad (1)$$
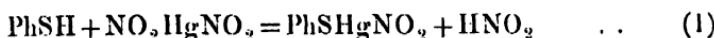
The nitrous acid thus liberated oxidises another pair of molecules of PhSH into diphenyldisulphide.

$$PhSH + O + HSPh = Ph-S-S-Ph + H_2O$$
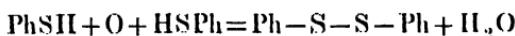
The latter compound now forms the following compound with mercuric nitrite :—

$$\begin{array}{ccc} PhS-SPh & & PhS-S-Ph \\ \wedge \longrightarrow & \blacktriangleright & \wedge \qquad + N_2O_3 \\ HgNO_2. NO_2 & & \end{array}$$
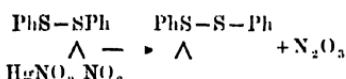
Two molecules of this compound again combine with a molecule of Ph_2S_2 (itself the product of oxidation of PhSH as stated above) and give rise to the compound,

$$\begin{array}{c} [2(Ph-S-SPh), Ph_2S_2] \\ \wedge \\ Hg-O \end{array}$$
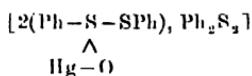
It may also be formulated as an "atomic" compound,

$$\begin{array}{cccc} & Ph & Ph & \\ & | & | & \\ Ph-S\!\!-\!\!-\!\!-\!\!S\!\!-\!\!-\!\!-\!\!S\!\!-\!\!-\!\!-\!\! & S-Ph \\ \wedge & | & | & \wedge \\ Hg-O & SPh & SPh & O-Hg \end{array}$$
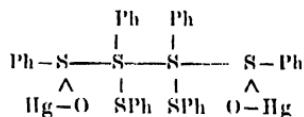
in which four sulphur atoms are tetravalent and two only divalent.

It may be objected here that as the compound $[3(PhS)HgO]_2$ is found to be free from phenyl-mercaptide-nitrite, the reaction as given above does not occur. It is, however, important to bear in mind that only the presence of a trace of nitrous acid is necessary to start the reaction ; when once it is set a-going there will always be sufficient quantities of it available, according to the reaction in stage (2). In fact, it is inconceivable that mercuric nitrite should act upon a mercaptan without giving rise simultaneously to a nitrite-mercaptide and nitrous acid.

The compound represented by the empirical formula Ph_2S_3Hg, is evidently a mercaptide and the mechanism of the reaction involved in its formation does not seem to offer a ready solution. The simplest explanation would appear to be as follows :—

$$\underset{Hg-O}{\overset{Ph-S-S-Ph}{\wedge}}$$

The oxygen atom of (see above) goes to oxidise two molecules of PhSH into PhSSPh. An atom of sulphur is now snatched away from the disulphide, which takes the place of oxygen producing

$$\underset{Hg\cdots S}{\overset{Ph-S-S-Ph}{\wedge}}$$

and the monosulphide Ph– S—Ph. It is the strong affinity of mercury for sulphur which supplies the motive of the reaction. (Cf. conversion of thiocarbamide into cyanamide by means of mercuric oxide).

Another mercaptan belonging to the heterocyclic series, viz., phenyl-methyl-thiohydantoin, gave the nitrite-mercaptide conforming to the formula

$$\begin{array}{c} CO-NPh \\ | \quad\quad | \\ | \quad CSHgNO_2 \\ | \quad\quad \| \\ CH_3CH-N \end{array}$$
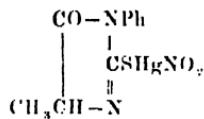
It is now generally admitted that it is a real mercaptan (*i.e.* ψ or pseudohydantoin) and not a hydantoin and this is further confirmed by the fact that by the interaction of mercuric nitrite no break-up of the complex molecule occurs, neither does the divalent sulphur become tetravalent by taking on ($-HgNO_2$) and ($-NO_2$) thus :—

$$R-S-HgNO_2 \longrightarrow \underset{HgNO_2\,NO_2}{\overset{R-S-HgNO_2}{\wedge}} \longrightarrow \underset{Hg-O}{\overset{R-S-HgNO_2}{\wedge}}$$
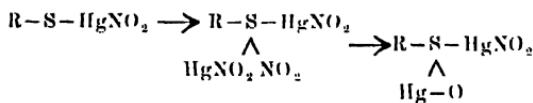
It has been invariably found that whenever a *potential* mercaptan is similarly treated, the reaction takes place, as indicated below.

POTENTIAL MERCAPTANS.

The designation of potential mercaptans has been given to those thio-compounds which by withdrawal of an atom of hydrogen from the neighbouring NH_2 or NH group behave as mercaptans. The interaction of mercuric nitrite with thiocarbamide as also its alkylated derivatives, thio-acetamide and thio-benzamide, has already been studied. The investigation has now been further extended to the

behaviour of this reagent towards the aryl substituted thiocarbamides, thiosemicarbazides, sulphocarbazides, etc.   It has been found that the reaction follows the ordinary course, but no detachment of the organic radical takes place.   Thus phenyl-thiocarbamide at first gives rise to :—

$$\begin{array}{c} NH \\ \parallel \\ C-SHgNO_2 \\ \mid \\ NHPh \end{array}$$
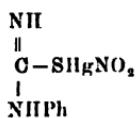
The divalent sulphur atom at once takes up an additional molecule of $Hg(NO_2)_2$ and becomes tetravalent :

$$\begin{array}{c} NH \\ \parallel \\ C\underline{\hspace{2cm}}S-HgNO_2 \\ \mid \qquad\quad \wedge \\ NHPh \quad HgNO_2 \; NO_2 \end{array}$$
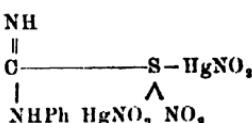
and ultimately the compound

$$\begin{array}{c} NH \\ \parallel \\ C\underline{\hspace{1cm}}S-HgNO_2 \\ \mid \qquad \wedge \\ NHPh \quad Hg-O \end{array}$$

is formed, with elimination of N_2O_3.

Sym. diphenyl thiourea or thiocarbanilide gives only on rare occasions the expected compound

$$\begin{array}{c} NPh \\ \parallel \\ PhNH-C-SHgNO_2, \\ \wedge \\ Hg-O \end{array}$$
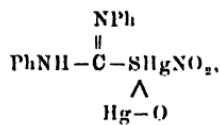
but the one which is generally obtained, conforms to the formula

$$\begin{array}{c} NPh \\ \parallel \\ C-SHgNO_2 \\ \mid \\ NhgPh \end{array}$$
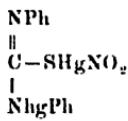
in the latter case, an atom of hydrogen of the adjacent imino-group is further replaced by (hg).

Thiosemicarbazide and its aryl substituted derivatives similarly yield

$$\begin{array}{c} NH(orPh) \\ \parallel \\ C-SHgNO_2 \\ \mid \\ NhgNH_2 \end{array}$$
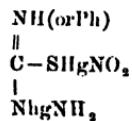
with the simultaneous displacement of an H atom of $NHNH_2$ by hg (Cf. guanidine in which two atoms of H are replaced by hg).

Diphenylsulphocarbazide also behaves similarly and gives the compound,

$$\begin{array}{l} \text{N}-\text{NHPh} \\ \| \\ \text{C}-\text{S}-\text{HgNO}_2 \\ | \\ \text{NhgNHPh} \end{array}$$
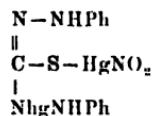
It is noteworthy, that as the molecules have already become highly loaded by the introduction of an additional (hg), the sulphur atom has lost the capacity of becoming tetravalent by further taking up (—HgNO$_2$) and (—NO$_2$). In connection with this it is important to remember that although thiocarbamide, as a rule, gives rise to the inorganic mercuri-sulphoxy-nitrite [3(SHg NO$_2$)HgO]$_2$, under special conditions it can be made to yield the compound

$$\begin{array}{c} \text{NH} \\ \| \\ \text{NH}_2-\text{C}-\text{S}-\text{HgNO}_2 \\ \wedge \\ \text{Hg}-\text{O} \end{array}$$
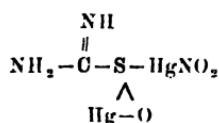
(l. c. p. 192). In other words, what was regarded as an exception or aberrant type of reaction, now becomes the rule.

Attention may also here be drawn to the interesting analogous case of phenyl dithiocarbazinic acid (or rather its K-salt), which combines in itself the function of a real as also a potential mercaptan.

Thus, PhNHNhCSSK $\xrightarrow{\text{Hg(NO}_2)_2}$ PhNH·N:C$<^{\text{SK}}_{\text{SH}}$ $\longrightarrow$ PhNH·N:C$<^{\text{SHgNO}_2}_{\text{SH}}$

$$\longrightarrow \text{PhNH·N:C}<^{\text{SHgNO}_2}_{\text{SHgNO}_2} \\ \wedge \\ \text{Hg}-\text{O}$$
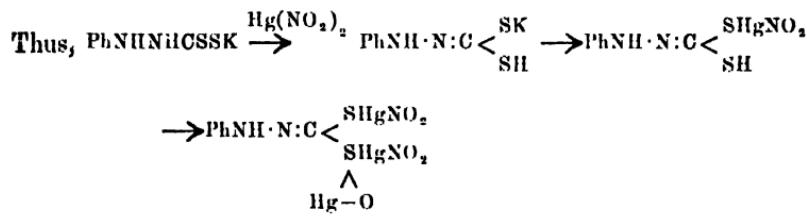
The sulphur atom belonging to the potential mercaptan alone becomes tetravalent.

It thus verifies the law enunciated above inasmuch as the two sulphur atoms behave differently.

Perhaps one of the most striking reactions in the present series is that which takes place between mercuric nitrite and p-chlorophenyl-thiocarbamide. A nitrite free from chlorine of the empirical formula $C_7N_4H_6Hg_3S_2O_5$ is invariably obtained. Its constitution is evidently represented by the formula

$$\text{O}_2\text{NHg}-\text{S}-<\boxminus>-\text{NH}-\text{C}\text{------}\text{SHgNO}_2 \\ \| \quad \wedge \\ \text{NH} \ \text{Hg}-\text{O}$$
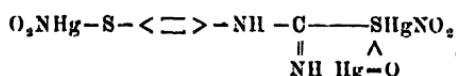
The mechanism of the reaction involved here is rather complicated.
The first stage evidently gives,

$$Cl < \equiv > -NH-C-SHgNO_2$$
$$\overset{\parallel}{NH}$$
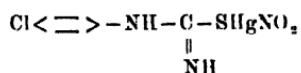
The chlorine in the altered condition of the molecule now becomes
active and thus can no longer retain its place, and parts company
with the parent body and in the nascent state acts upon a second
molecule of the reaction product

$$ClC_6H_4-NH-C-\overset{\vdots}{SHgNO_2}$$
$$\overset{\parallel}{NH}$$
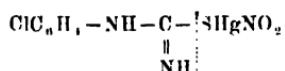
and brings about its disruption as indicated by the dotted line; while
the stable radical ($-SHgNO_2$) occupies the place of the halogen
atom, the sulphur atom which is attached to the carbon atom of the
thiocarbamide becomes tetravalent, as in the compounds described
immediately above.

The constitution proposed is supported by the product it yields with
ethyl iodide when the split takes place as indicated by the dotted line,

$$< \equiv > -NH-C-NH$$
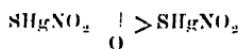
$$SHgNO_2 \quad \overset{\vdots}{O} > SHgNO_2$$

Both the radicals ($-SHgNO_2$) get detached from the parent body
and the expected reaction takes place :—

$$\begin{matrix} -SHgNO_2 \\ Et \end{matrix} \quad = -SHgI + EtNO_2$$
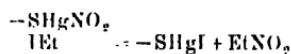
($-SHgI)_2$, it will be remembered, is also the main product of
the interaction between [3 (SHg NO$_2$) HgO]$_2$ and alkyl iodide. At
the same time [Et$_2$S$_2$, HgI$_2$, EtI] is also formed (Cf. Trans. 1917,
111, p. 108.)

MIGRATION OF ALKYL RADICALS.

The behaviour of mercuric nitrite towards compounds of the type
represented by Ph-imido-phenyl-carbamic-thiomethylester is equally
striking. The latter is a derivative of pseudo (ψ) thiocarbamide and
is assigned the formula

$$\begin{matrix} PhNH \\ | \\ CSMe \\ \parallel \\ PhN \end{matrix}$$

When it is treated with the reagent in question the original mercaptan is reproduced, the radical methyl being pushed back to the neighbouring nitrogen atom, hydrogen taking its place, thus :—

$$
\begin{array}{ccc}
\text{PhNH} & \text{PhN(Me)} & \text{PhN(Me)} \\
| & | & | \\
\text{CSMe} \xrightarrow{\ \text{Hg(NO}_2)_2\ } & \text{C} - \text{SH} \longrightarrow & \text{C} - \text{S} - \text{HgNO} \\
\| & \| & \| \\
\text{PhN} & \text{PhN} & \text{PhN}
\end{array}
$$
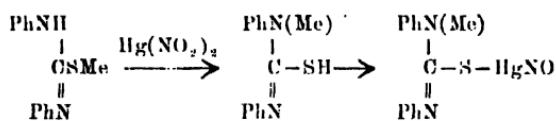
In fact nitrous fumes are evolved during the reaction. It is the marked affinity of the radical $(-\text{HgNO}_2)$ for the sulphur atom which brings about the notable displacement of methyl radical. Additional instances will be furnished below (pp. 10—13).

That the reactions described above are of wide applications are borne out by the deportment of methyl ether of phenyl dithiobiazoline sulphydrate. With mercuric-nitrite it gives the following compound :—

$$
\begin{array}{l}
\quad\quad\quad\text{CH}_3 \\
\quad\quad\quad\ | \\
\text{PhN} - \text{N} \\
\quad\quad\ \ \ | \\
\quad\quad\quad\quad \text{C} - \text{SHgNO}_2 \\
\quad\quad\quad \bigwedge \\
\text{SC} - \text{S}\quad\text{HgNO}_2
\end{array}
$$
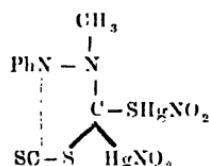
Here by the shifting of the radical (CH_3) to the neighbouring nitrogen atom, a bond of carbon indicated by the thick line is released and this is satisfied by an additional radical (HgNO_2). It might be objected that there is no instance as yet known in which (HgNO_2) group is directly attached to a carbon atom. Another alternative explaration might be offered to the above reaction, viz., that the divalent sulphur atom becomes tetravalent by attaching to itself two (HgNO_2) groups, thus :—

$$
\begin{array}{ll}
\text{Ph} - \text{N} - \text{N} & \text{HgNO}_2 \\
\quad\quad \| \quad\ | \\
\quad\quad \text{C} - \text{S} - \text{CH}_3 \\
\quad\quad\ | \quad\ | \\
\quad \text{SC} - \text{S} & \text{HgNO}_2
\end{array}
$$
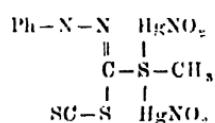
Last of all, may be described the behaviour of the above reagent towards a complex disulphide, viz., Ph-dithiobiazoline disulphide. The reaction takes place as in the case of ethyl disulphide (Trans. 1916, *109* 133) and the following compound is formed :—

$$
\begin{array}{l}
\text{PhN} - \text{N}\quad\text{HgNO}_2\ \ \text{NO}_2\quad\text{N} - \text{NPh} \\
\quad\ \| \quad\ | \quad\quad\ | \quad\quad\ \| \\
\quad\ \text{C} - \text{S} - - - \text{S} - - - \text{C} \\
\quad\ | \quad\ | \quad\quad\ | \quad\ | \\
\text{SC} - - \text{S}\quad\text{Hg} - - - \text{O}\quad\text{S} - \text{CS}
\end{array}
$$
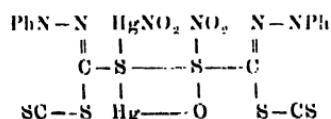
INTERACTION WITH THE ALKYL IODIDES:
FORMATION OF MONO, DI- AND TRI-SULPHONIUM COMPOUNDS.

The reaction follows the general course with this material differ-
ence that the complex radical being over-weighted can no longer
retain its entity but the less durable part of it is usually shattered.
Of special significance from this point of view is the rupture of the
ring of the heterocyclic mercaptide nitrites. Let us take the specific
case of dithiobiazoline derivative. When digested with an alkyl
iodide we should expect $[RR'S_2, HgI_2, R'I]$ *i.e.* in this case.

$$\begin{array}{l} /PhN-N\\ \qquad\qquad C\\ SC-S/ \end{array}\Big| R'S_2, HgI_2, R'I \qquad\qquad \begin{array}{l} PhN-N \quad gI \; R'\\ \;\;|\quad\;\; \|\quad\;\; |\quad\; |\\ \;\;\;\;\;\; C-S \;-\; S--R'\\ \;\;\;\;\;\; |\quad | \quad |\\ \;\;\overline{|\quad\;\;|}\;\diagdown I \quad I\\ \;\;SC-S \end{array}$$
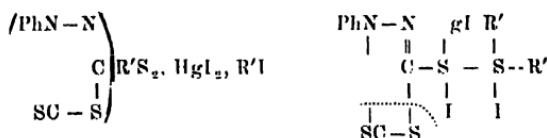
But a scission now takes place as shown by the dotted line and the
disulphonium compound,

$$\begin{array}{c} \qquad\;\; HgI \; R'\\ \qquad\;\;\;\; | \quad |\\ PhN-N \cdot C-S-S-R'\\ \;\;\;| \qquad\quad | \quad | \quad |\\ \qquad\qquad\quad I \quad I \end{array}$$
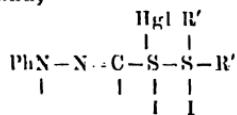
is potentially formed, the unsaturated atoms C and N add on R'
from the interacting R'I and the product really formed is

$$\begin{array}{cc} & \qquad\qquad HgI \; R'\\ Ph & \qquad\qquad\;| \quad\; |\\ \quad>N-N=C-S-S-R' & \qquad (I)\\ R' & \qquad\;\; | \quad | \quad |\\ & \qquad\;\; R' \quad I \quad I \end{array}$$
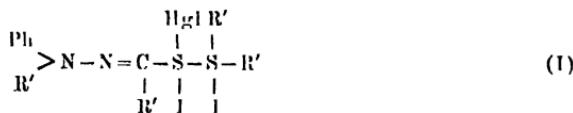
By further action of R'I the phenyl group is replaced by R' and
the following compound is formed :

$$\begin{array}{cc} & \qquad\quad HgI \; R'\\ & \qquad\quad\; | \quad\; |\\ R'-N-N \cdot \; C-S-S-R' & \qquad (II)\\ \;\;\;| \qquad\quad | \quad | \quad |\\ \;\;\;R' \qquad\quad R' \quad I \quad I \end{array}$$
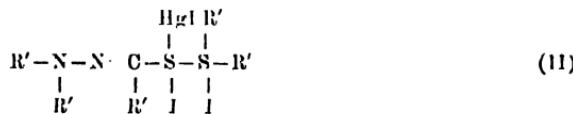
Finally, the double bond between C and N is removed, two more
R''s being taken up and we get

$$\begin{array}{cc} & \qquad\quad R' \; HgI \; R'\\ R' & \qquad\quad | \quad\; | \quad\; |\\ \quad>N-N-C-S-S-R' & \qquad (III)\\ R' & \qquad\; | \quad | \quad | \quad |\\ & \qquad\; R' \quad R' \quad I \quad I \end{array}$$
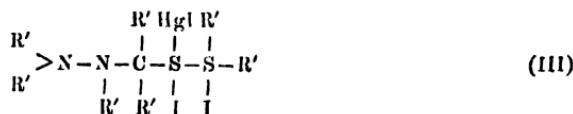
By the action of methyl iodide, two compounds corresponding
to stages I and III have been isolated, whereas, in the case of
ethyl iodide only one compound has been isolated which corresponds
to stage II as shown above.

As a matter of fact, it has been observed that the yield of the compound with methyl iodide, corresponding to stage I decreases with the lengthening of the period of reaction. So, it is evident that it is only an intermediate product which is gradually transformed into the compounds corresponding to stages II and III, which undergo no further change.

It has already been shown that these disulphonium-mercuri-iodides can be directly synthesised from their components (l. c. 611). Dithiobiazoline-disulphide dissolved in carbon-disulphide was refluxed with ethyl iodide and mercuric iodide. In this case, not only was there no rupture of the dithiobiazoline radicals, but in one instance one of the tertiary nitrogen atoms became quaternary by taking up a molecule of ethyl iodide and the compound which was obtained corresponded to the formula :—

$$
\begin{array}{c}
\text{Et I} \\
\vee \\
\text{PhN} - \text{N HgI Et N} - \text{NPh} \\
\parallel \quad | \quad | \quad \parallel \\
\text{C} - \text{S} - \text{S} - \text{C} \\
| \quad | \quad | \quad | \\
\text{SC} - \text{S} \quad | \quad | \quad \text{S} - \text{CS}
\end{array}
$$
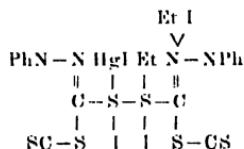
It is thus evident that the rupture of the complex radical can only take place when it separates out in the *nascent* state, and is therefore in a state of strain and is endowed with vibratory motion.

$$
\begin{array}{c}
\text{RS}_2\text{HgNO}_2 \\
\text{I Et} \\
\text{RS HgNO}_2 \qquad (\text{l.c. 603}) \\
\text{I Et}
\end{array}
$$
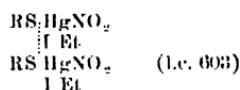
This is what might have been expected. In the ready made molecule the component radical is inert and sluggish and naturally does not lend itself to disruptive influences. On the other hand when dithio-biazoline-mercaptide-nitrite is treated with an alkyl iddide, it suffers a split and its stable portion alone takes part in the formation of the disulphonium compound.

The reaction of alkyl iodides on phenyl-mercuri-mercaptide-nitrite follows the usual course and yields compounds of the general formula $[\text{PhR}'\text{S}_2, \text{HgI}_2, \text{R}'\text{I}]$. The compound obtained from $\text{Ph}_2\text{S}_3\text{Hg}$ and alkyl iodide is what was actually anticipated. It contains all the sulphur atoms in a single chain and all of them are

quadrivalent. The reaction can be explained in the following
way :—

In the first stage the bond between Hg and S as shown by the
dotted line is severed.

$$Ph-S-S-Ph \qquad \overset{|\quad|}{Ph-S-S-P}$$
$$\overset{\wedge}{} \quad \longrightarrow$$
$$S-Hg \qquad\qquad -S-Hg-$$
$$\qquad\qquad |$$
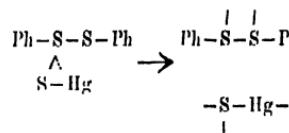
This assumed intermediate product in presence of an excess of
alkyl iodide acquires the power of transforming all of its sulphur
atoms into the quadrivalent form and the unsaturated bonds are
simply saturated by the iodine atoms and alkyl groups,

$$
\begin{array}{c}
\text{Ph Ph} \\
|\ \ |\ \ |\quad CH_3I \\
-S-S-S-\ \longrightarrow CH_3-S-S-S-CH_3 \\
|\ \ | \\
Hg
\end{array}
\qquad
\begin{array}{c}
|\quad\text{Ph Ph} \\
|\ \ |\ \ | \\
\\
HgI\ |\ \ |
\end{array}
$$
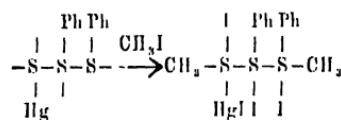
and thus a *trisulphonium* compound is formed.

An interesting monosulphonium compound has been obtained
from the mercaptide nitrite of thiocarbanilide, *viz.*,

$$
\begin{array}{l}
NPh \\
\| \\
C\!\!-\!\!\!-\!\!\!-\!\!\!- S-HgNO_2 \\
|\qquad \wedge \\
NHPh\ Hg-O
\end{array}
$$
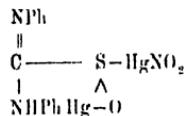
in which the sulphur atom is already tetravalent. It cannot and in
fact does not show any tendency to give a disulphonium compound
by doubling the formula (or rather by doubling the sulphur chain).
The reaction probably takes place as shown below :—

$$
\begin{array}{l}
NPh \\
\| \\
C\!\!-\!\!\!-\!\!\!-\!\!\!-S-HgNO_2 \\
|\qquad \wedge \\
NHPh\ Hg-O
\end{array}
\ \overset{EtI}{\longrightarrow}\
\begin{array}{l}
Hg- \\
\|\ \ | \\
C-S- \\
|\ \ | \\
\end{array}
\ \longrightarrow\
\begin{array}{l}
Et\ HgI \\
|\ \ | \\
I-C-S-I \\
|\ \ | \\
Et\ Et
\end{array}
$$
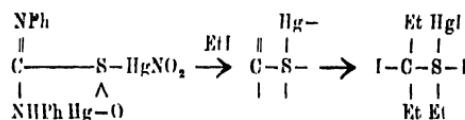
Here both the phenylamino-groups are replaced by ethyl.

The interaction of mercaptide-nitrite of phenyl-imidophenyl-
carbamic-thio-alkylesters with alkyl iodide takes place as in the
case of the simple mercaptide nitrites forming compounds of
the type $[R_2S_2, HgI_2, R'I,]$ and in fact the reaction advances

a step further, both of the complex groups being replaced by simple alkyl ones.

$$
\begin{array}{ccccc}
\text{PhNR} & \text{PhNR} \; \text{HgI R}' & & \text{RNPh} & \text{HgI R}' \\
| & | \quad\quad | \quad | & & | & | \quad | \\
\text{C-SHgNO}_2 \longrightarrow & \text{C}\;|\!-\!\text{S}\!-\!\text{S}\!-\!| \; \text{C} & \longrightarrow & \text{R}'\!-\!\text{S}\!-\!\text{S}\!-\!\text{R}' \\
\| & \| \quad\quad | \quad | \; \| & & | \quad | \\
\text{PhN} & \text{PhN} \quad\quad \text{NPh} & &
\end{array}
$$
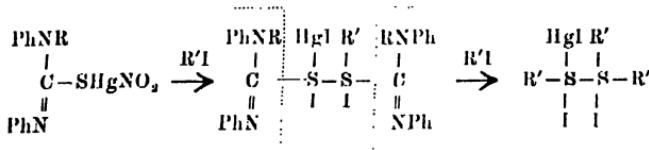
It may be added here in this connection that the formation of this type of disulphonium compounds proves conclusively the hypothesis of pushing back (migration) of alkyl groups to the adjacent amino- or imino-groups in the formation of the mercaptide nitrites ; the (—HgNO$_2$) groups remain attached to the sulphur and not to the neighbouring nitrogen (vide *ante* p 8). For, otherwise the formation of the corresponding disulphonium compounds is inexplicable.

It has been shown in the previous communication that [3(SHgNO$_2$)HgO]$_2$ alone, by interacting with alkyl iodides, gives disulphonium compounds of the type [R$_2$S$_2$, HgI$_2$, RI] (T. 1917, *111*, 108).

So, the presence of the group (--SHgNO$_2$) is essential to the formation of disulphonium compounds, whether it be present as such as in [3(SHgNO$_2$)HgO]$_2$ or associated with simple alkyl groups as in EtSHgNO$_2$ or in complex groups as in

$$
\begin{array}{c}
\text{PhN}-\text{N} \\
|\quad\;\; \| \\
|\quad\;\; \text{C}-\text{S}-\text{HgNO}_2 \\
|\quad\;\; | \\
\text{SC}-\text{S}
\end{array}
$$
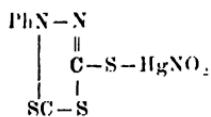
EXPERIMENTAL

Interaction of N-Phenyl-Dithiobiazoline Sulphydrate and Mercuric Nitrite.

As it was difficult to obtain the mercaptan itself in a state of purity the K-salt was (Ber 27. 2516) treated with mercuric nitrite. It had the additional advantage inasmuch as no nitrous acid was liberated, which would have an oxidising action on the one or more atoms of sulphur. The mercury salt was obtained sometimes anhydrous, but often combined with three, five and eight molecules of water ; the degree of hydration, evidently depending upon the dilution of the reacting substances, the anhydrous variety had a deep yellow colour while the compound containing the largest number of water molecules (*viz.* eight) was almost white. They all evolved

nitrous fumes when treated with hydrochloric acid. It may be added here once for all, that these as also the following compounds were always dried in a vacuum over sulphuric acid.

Results of analysis:—

0·2724 gave 0·1187 Hg; Hg = 43·59.

0·1100 gave 0·0810 CO_2 and 0·0187 H_2O; C = 20·08; H = 1·89.

0·1420 gave 10·2 c.c. N_2 at 25°c and 760 mm; N = 8·10.

Calc. for $C_8H_5N_3S_3HgO_2$: Hg = 42·46; C = 20·38; H = 1·06; N = 8·92.

*The above with 3 H_2O:—

found Hg = 38·05; C = 18·28; H = 3·29; N = 8·30; S = 18·56.

Calc. Hg = 38·09; C = 18·29; H = 2·1; N = 8·00; S = 18·29.

The compound with 5 H_2O:—

found Hg = 35·75; C = 16·92; H = 3·20.

Calc. Hg = 35·65; C = 17·12; H = 2·67.

The compound with 8 H_2O:—

found Hg = 32·10; C = 15·10; H = 3·56; N = 6·16; S = 15·08.

Calc. Hg = 32·52; C = 15·63; H = 3·41; N = 6·83; S = 15·61.

Mercuric nitrite and thiobiazole-disulphydrate :

As a rule the product which is the outcome of the condensation of three molecules of thiobiazole-disulphydrate is formed. It is only in rare cases that the condensation with four molecules takes place. As in the case of the previous compounds different number of water molecules are associated with the molecule, the degree of hydration depending upon the concentration of the reacting substances. The product formed by condensation of three molecules, *viz.*,

$$\begin{array}{c} S-(C_2N_2S)_3S_4-S \\ | \qquad\qquad | \\ Hg\text{———}O\text{———}Hg \end{array}$$, 8H_2O gave the following results on analysis:—

0·1370 gave 0·0639 HgS and 0·2749 $BaSO_4$; Hg = 40·21, S = 27·56.

0·1229 gave 0·0383 CO_2; C = 8·50.

0·1377 gave 10·2 c.c. N_2 at 30°c and 760 mm; N = 8·18.

Calc. Hg = 39·85; S = 28·69; C = 7·17; N = 8·37.

---

* In order to economise space details of the results of analysis of the hydrated varieties are not given. It is necessary to point out here also that the percentage of hydrogen in most cases comes out unusually high as traces of mercurial vapour are apt to be carried into the $CaCl_2$ tube.

The above with $5H_2O$:—

0·2010 gave 0·0981 HgS and 0·4421 $BaSO_4$; $Hg = 42·07$; $S = 30·21$.

0·1592 gave 13·36 c.c. N_2 at 32°c and 760 mm ; $N = 9·36$

Calc. $Hg = 42·11$; $S = 30·31$; $N = 8·84$.

The above with $2H_2O$:—

0·1392 gave 0·0732 HgS and 0·3182 $BaSO_4$; $Hg = 45·33$; $S = 31·40$.

0·1865 gave 16 c. c. of N_2 at 29°c and 760 mm ; $N = 9·52$.

Calc. $Hg = 44·65$; $S = 32·14$; $N = 9·38$.

The compound formed by condensation of four molecules, *viz.*,

$$\begin{array}{c} S-(C_2N_2S)_4S_6-S \\ | \qquad \qquad | \\ Hg \text{——} O \text{——} Hg \end{array}, +3H_2O$$ gave the following results of analysis:—
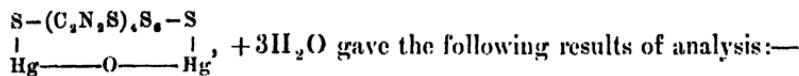
0·2800 gave 0·1064 Hg and 0·7654 $BaSO_4$; $Hg = 37·94$; $S = 37·54$

0·0904 gave 9·4 c.c. N_2 at 30°C and 760 mm ; $N = 11·48$

Calc. $Hg = 37·67$; $S = 36·16$; $N = 10·55$

Interaction of mercuric nitrite with the potassium salt of thiobiazole-disulphydrate :

When the reaction is made to take place with the freshly dissolved K-salt in water, the compound

$$\begin{array}{c} N-N \\ \| \quad \| \\ KSC \quad C-SHgNO_2 \\ V \\ S \end{array}$$
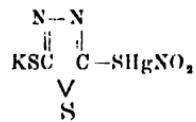
is formed. The product is a greenish yellow nitrite and gave the following results on analysis:—

0·3037 gave 0·1714 HgS and 0·5702 $BaSO_4$; $Hg = 48·65$ $S = 25·79$

0·1658 gave 12·6 c.c. N_2 at 31° and 760 mm ; $N = 8·37$

Calc. $Hg = 46·19$; $S = 22·17$; $N = 9·70$

The presence of potassium was also proved. The salt thus approaches almost the theoretical formula given above; but it is necessarily contaminated with a slight admixture of the next compound.

On keeping, the potassium salt undergoes hydrolysis in aqueous solution as is proved by its alkalinity. In two preparations greenish-yellow precipitates were obtained.

The compound conforms to the formula

$$\begin{array}{ccc} S-(C_2N_2S)_2S_2-S \\ | \qquad\qquad | \\ Hg\text{------}O\text{------}Hg \end{array}$$
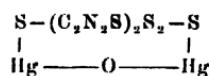
with a slight admixture of prep. I.

0·1626 gave 0·0986 HgS and 0·2624 BaSO$_4$; Hg = 52·28; S = 22·35

0·1620 gave 10·8 c.c. N$_2$ at 32°c and 760 mm; N = 7·36

Calc. for the pure compound, Hg = 56·i2; S = 26·97; N = 7·86

Mercuric nitrite and thiophenol :

Three different compounds have been isolated in this case. When the alcoholic solution of thiophenol is added very slowly to an excess of mercuric nitrite solution the reaction takes the ordinary course and PhSHgNO$_2$ is mainly formed. It is a dull yellow, light granular powder and is a real nitrite. But, when the thio-compound is in excess and the addition does not take place very slowly, the whole of the reaction mixture assumes a dirty yellow colour and nitrous fumes are evolved profusely. On standing, however, the reaction subsides and a white granular powder is obtained. It can be crystallised from hot benzene and when absolutely pure has a sharp m. p. 110°. Under slightly varying conditions an oxy-compound [(PhS)$_3$HgO]$_2$] is formed. As is evident, it is not easy to control the reaction so as to give one product to the complete exclusion of the others. The nitrite-mercaptide is always found to be admixed with the other products of this reaction.

The nitrite-mercaptide gave different results of analysis on different occasions, depending upon the proportion of Ph$_2$S$_3$Hg admixed with it; generally, however, the values have been found to be intermediate between that of the pure compound and Ph$_2$S$_3$Hg.

	Calc. for PhSHgNO$_2$	found	Calc. for Ph$_2$S$_3$Hg.
Hg =	56·33	51·30	44·44
S =	9·01	14·23	21·30
C =	20·28	26·44	32·00
H =	1·41	2·24	2·30

It will be noticed that the substance consisted of almost equal proportions of both.

The compound of the formula Ph_3S_3Hg gave the following results on analysis :—

0·1942 gave 0·0864 Hg and 0·3134 $BaSO_4$; Hg=44·48; S=22·16.

0·0964 gave 0·116 CO_2 and 0·0230 H_2O; C=32·17; H=2·7.

Calc. Hg=44·44; S=21·30; C=32·00; H=2·30.

The compound $[3(PhS)HgO]_2$ gave the following results on analysis :—

0·4016 gave 0·1506 Hg ; Hg=37·50

0·4333 gave 0·5620 $BaSO_4$; S=17·82

0·1230 gave 0·1886 CO_2 ; C=41·81

Calc. Hg=38·02 S=18·25 ; C=41·07

The above two compounds were proved to be non-nitrogenous by combustion analysis.

Mercuric nitrite and phenyl-methyl-thiohydantoin : formation of the compound :

$$CO-NPh$$
$$|$$
$$CSHgNO_2 +2H_2O$$
$$\|$$
$$CH_3CH-N$$
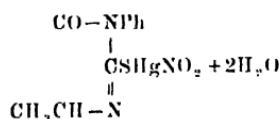
The compound was greenish-grey in colour and was a real nitrite.

0·1981 gave 0·0954 HgS and 0·1110 $BaSO_4$; Hg=44·51; S=7·69

0·1774 gave 0·1586 CO_2 and 0·0582 H_2O; C=24·38; H=3·65.

0·2836 gave 20·8 c. c. N_2 at 30° and 760 mm ; N=8·10

Calc. Hg=44·07 ; S=6·57 ; C=24·64 ; H=1·85 ; N=8·63.

POTENTIAL MERCAPTANS.

Mercuric nitrite and phenylthiourea : formation of the compound :

$$NH \quad HgNO_2$$
$$\| \quad |$$
$$C---S-Hg$$
$$| \quad \vee$$
$$NHPh \quad O$$
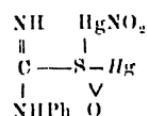
In this case the tendency to form into a pasty mass is well marked and it was with great difficulty and after several failures that the pure product was obtained. It was deep yellow in colour and was a nitrite.

0·3512 gave 0·2350 Hg and 0·1227 $BaSO_4$; Hg=66·9 ; S=4·8

0·2015 gave 11·4 c. c. N_2 at 26° and 760 mm ; N=6·40

Calc. Hg=65·25 ; S=5·22 ; N=6·85.

Mercuric nitrite and sym-diphenyl thiourea; formation of the] compound :

$$\begin{array}{c} \text{NHPh} \\ | \\ \text{PhN}=\text{C}-\text{S}-\text{HgNO}_2 \\ \wedge \\ \text{Hg}-\text{O} \end{array}$$
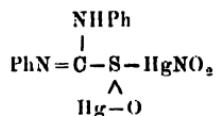
It was a brownish yellow granular powder, and was a nitrite.

0·3422 gave 0·1970 Hg and 0·1069 $BaSO_4$; Hg=57·57 ; S=4·64

0·3970 gave 0·1141 CO_2 and 0·0244 H_2O ; C=22·27 ; H=1·94

Calc. Hg=58·07 ; S=4·64; C=22·64; H=1·60.

The above compound is only rarely formed. The product generally obtained conforms to the formula

$$\begin{array}{c} \text{NhgPh} \\ | \\ \text{PhN}=\text{C}-\text{SHgNO}_2 \end{array}$$
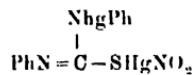
0·1555 gave 0·0947 HgS and 0·0741 $BaSO_4$; Hg=52·51 ; S=6·55

0·2275 gave 0·1367 HgS and 0·0992 $BaSO_4$; Hg=51·80 S=5·99

Calc. Hg=52·43; S=5·59.

Mercuric nitrite and thiosemicarbazide: formation of the compound :

$$\begin{array}{c} \text{NhgNH}_2 \\ | \\ \text{NH}-\text{C}-\text{SHgNO}_2 \end{array}$$
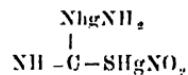
The thio-compound was taken in aqueous solution. The product was a dull yellow granular powder and was a nitrite.

0·2715 gave 0·1870 HgS and 0·1362 $BaSO_4$; Hg=68·87 ; S=6·89

0·1032 gave 11·8 c. c. N_2 at 32°c and 760 mm ; N=12·6

Calc. Hg=68·96; S=7·36 : N=12·87.

Mercuric nitrite and phenyl thiosemicarbazide: formation of the compound :

$$\begin{array}{c} \text{NhgNHPh} \\ | \\ \text{NH}=\text{C}-\text{SHgNO}_2 \end{array}$$
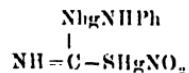
It is an orange yellow granular powder and is a nitrite.

0·30º1 gave 0·1824 Hg and 0·1326 $BaSO_4$; Hg=59·2 ; S=5·91

0·2036 gave 20·5 c. c. N_2 at 33° and 760 mm ; N=11·10

Calc. Hg=58·70; S=6·26 ; N=10·96.

Mercuric nitrite and diphenyl thiosemicarbazide: formation of the compound:

$$Nhg \cdot NHPh$$
$$|$$
$$PhN = C - SHgNO_2$$
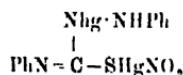
It is an orange yellow granular powder and is a nitrite.

0·1797 gave 0·0925 Hg; Hg=51·48.

0·2577 gave 0·2498 CO_2 and 0·0523 H_2O; C=26·24; H=2·25.

0·1380 gave 11·7 c. c. N_2 at 32° and 760 mm; N=9·38.

Calc. Hg=51·11; C=28·57; H=1·87; N=9·54.

Mercuric nitrite and diphenyl sulpho-carbazide: formation of the compound:

$$Nhg \cdot NHPh$$
$$|$$
$$PhNH \cdot N = C - SHgNO_2$$
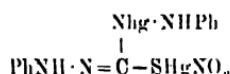
It is a beautiful pink coloured granular powder and is a nitrite.

0·1667 gave 0·0965 HgS and 0·0607 $BaSO_4$; Hg=49·91; S=5·00

0·1463 gave 0·1404 CO_2 and 0·0327 H_2O; C=26·14; H=2·48.

0·1167 gave 11·7 c. c. N_2 at 30° and 760 mm; N=11·26.

Calc. Hg=49·83; S=5·31; C=25·91; H=1·82; N=11·62.

Mercuric nitrite and phenyl dithiocarbazinic acid phenylhydrazine: formation of the compound:

$$S - HgNO_2$$
$$|$$
$$PhNH \cdot N : C - SHgNO_2$$
$$\wedge$$
$$Hg - O$$
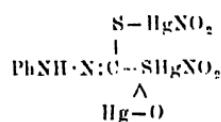
It is a blackish violet granular powder and is a nitrite.

0·1971 gave 0·1549 Hg and 0·1031 $BaSO_4$; Hg=67·71; S=7·18

0·1651 gave 8·5 c. c. N_2 at 27° and 760 mm; N=5·89.

Calc. Hg=67·40; S=7·19; N=6·30; C=9·44.

Mercuric nitrite and K-phenyldithiocarbazinate: formation of the same compound as above:

0·3294 gave 0·2239 Hg; Hg=67·97.

0·1790 gave 0·0607 CO_2 and 0·0158 H_2O; C=9·24; H=0·98

0·1321 gave 6·8 c. c. N_2 at 27° and 760 mm. N=5·88.

Calc. see above.

Mercuric nitrite and p. chloro-phenyl-thiourea : formation of the compound :

$$NO_2HgS-<\;\underline{\underline{\;\;}}\;>-NH$$
$$|$$
$$HN=C-S-HgNO_2$$
$$\wedge$$
$$Hg-O$$
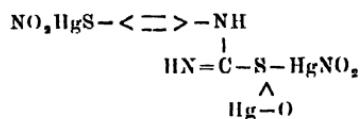
Great caution is necessary in the preparation of this compound. An acetone solution of the thio-compound is added in a thin stream under vigorous shaking to a solution of mercuric nitrite, care being taken that the latter is always in *large excess ;* otherwise, the product turns blackish, being slowly converted into mercuric sulphide. The compound was yellow and sometimes orange red and responded to the tests of a nitrite.

0·3578 gave 0·2410 Hg ; Hg.=67·37.

0·1823 gave 0·0961 BaSO$_4$; S=7·24.

0·1302 gave 0·0442 CO$_2$ and 0·0192 H$_2$O ; C=9·26 ; H=1·64.

0·1448 gave 7·4 c. c. N$_2$ at 20° and 760 mm ; N=5·87.

Calc. Hg=67·42 ; S=7·19 ; C=9·44 ; H=0·67 ; N=6·29.

Absence of chlorine was also proved.

MIGRATION OF ALKYL RADICALS.

Mercuric nitrite and imido-phenyl-carbamic-thio-methylester : formation of the compound :

$$NCH_3$$
$$\|$$
$$C-SHgNO_2$$
$$|$$
$$NHPh$$
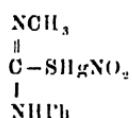
It is formed as a yellowish white flocculent precipitate which when dry turns orange yellow.

0·3050 gave 0·1464 Hg and 0·1533 BaSO$_4$; Hg=48·01 ; S=6·91

0·1461 gave 0·1238 CO$_2$ and 0·0329 H$_2$O ; C=23·10 ; H=2·51

Calc. Hg=48·42 ; S=7·75 ; C=23·24 ; H=2·18.

Mercuric nitrite and imido-phenyl-carbamic-thioethylester : formation of the compound :

$$NC_2H_5$$
$$\|$$
$$C-SHgNO_2$$
$$|$$
$$NHPh$$
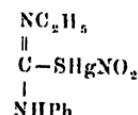
0·2918 gave 0·1340 Hg ; Hg = 45.93 ;

0·1936 gave 17·10 c. c. N_2 at 28° and 760 mm ; N = 9·92.

Calc. Hg = 46·84 ; N = 9·97.

Mercuric nitrite and phenyl imidophenyl carbamicthiomethylester: *formation of the compound :*

$$\begin{array}{l} NPh \\ \parallel \\ C-SHgNO_3 \\ \mid \\ NMePh \end{array}$$
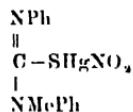
It is an orange yellow granular powder and is a nitrite.

0·2801 gave 0·1038 Hg ; Hg = 39·86.

0·2377 gave 0·3047 CO_2 and 0·0738 H_2O; C = 34·70 ; H = 3·45.

0·2863 gave 22·6 c. c. N_2 at 29°c and 760 mm. ; N = 8·99.

Calc. Hg = 40·9 ; C = 34·50 ; H = 2·67 ; N = 8·86.

Mercuric nitrite and phenylimidophenyl carbamicthioethylester: *formation of the compound :*

$$\begin{array}{l} NPh \\ \parallel \\ C-SHgNO_2 \\ \mid \\ NEtPh \end{array}$$
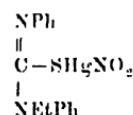
It is an orange yellow granular powder and is a nitrite.

0·3730 gave 0·1478 Hg and 0·1874 $BaSO_4$; Hg = 39·63 ; S = 6·91.

0·1232 gave 9·4 c. c. N_2 at 32° and 760 mm. N = 8·43.

Calc. Hg = 39·92 ; S = 6·39 ; N = 8·38.

Mercuric nitrite and methyl ether of ph. dithiobiazoline sulphydrate: *formation of the compound :*

$$\begin{array}{l} CH_3 \\ \mid \\ Ph\cdot N-N \\ \mid \\ C-S-HgNO_2 \\ \wedge \\ SC-S\,HgNO_2 \end{array}$$
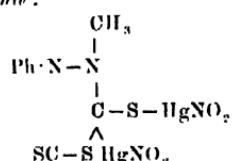
The thio-compound was dissolved in chloroform and the clear solution thus obtained was vigorously agitated with mercuric nitrite solution for nearly half an hour, when an emulsion was formed. It was allowed to stand over-night. A cream coloured granular mass was obtained, which was thrown on filter and washed with water, alcohol

and finally with chloroform and dried in a vacuum desiccator. It evolved nitrous fumes with HCl.

0·1984 gave 0·1055 HgS and 0·1384 BaSO$_4$; Hg = 45·85; S = 9·58.

0·1879 gave 0·0884 CO$_2$ and 0·0296 H$_2$O; C = 12·83; H = 1·75.

0·1603 gave 10·2 c. c. N$_2$ at 30° and 760 mm; N = 7·03.

Calc. Hg = 45·67; S = 10·96; C = 12·33; H = 0·91; N = 6·39.

Mercuric nitrite and phenyldithiobiazoline disulphide: formation of the compound:

$$\begin{array}{ccccc} \text{PhN}-\text{N} & \text{O} & ---\text{Hg} & \text{N}-\text{NPh} \\ | & \parallel & | & | & \parallel \\ | & \text{C}---\text{S} & ----\text{S}---\text{C} \\ | & | & | & | & | \\ \text{SC}-\text{S} & \text{HgNO}_2 & \text{NO}_2 & \text{S}-\text{CS} \end{array}$$
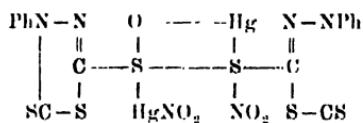
0·2940 gave 0·1288 Hg; Hg = 43·80.

0·1190 gave 9·00 c. c. N$_2$ at 30° and 760 mm; N = 8·35.

Calc. Hg = 41·75; N = 8·77·

INTERACTION WITH THE ALKYL IODIDES.

Mercaptide nitrite of n-phenyl-dithiobiazoline sulphydrate and methyl iodide: formation of the compound:

$$\begin{array}{cc} \text{CH}_3 & \text{CH}_3 \; \text{HgI} \; \text{CH}_3 \\ & | \quad | \quad | \\ >\text{N}-\text{N}--\text{C}---\text{S}---\text{S}-\text{CH}_3 \\ \text{CH}_3 & | \quad | \quad | \quad | \\ & \text{CH}_3 \; \text{CH}_3 \; \text{I} \quad \text{I} \end{array}$$
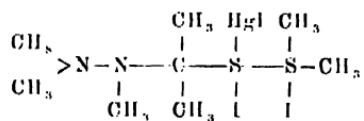
The method of procedure is exactly the same as in the interaction of mercury-mercaptide nitrites and the simple alkyl iodides. After refluxing with methyl-iodide it was found that a portion was left undissolved, which, when purified by washing several times with acetone, melted sharply at 127 °C. The soluble portion (in acetone) however, had to be purified by precipitation by addition of ether, and the pure compound thus obtained melted sharply at 107°. It gave the following results on analysis :—

0·3129 gave 0·0811 Hg; Hg = 25·92.

0·4107 gave 0·1066 Hg and 0·3718 AgI; Hg = 25·96; I = 48·72.

0·0844 gave 0·0363 CO$_2$ and 0·0237 H$_2$O; C = 11·73; H = 2·45.

C$_8$H$_{21}$N$_2$S$_2$HgI$_3$ requires, Hg = 25·32; I = 48·23; C = 12·15; H = 2·66.

The insoluble compound (in acetone) of the formula

$$
\begin{array}{c}
\text{Ph} \\
>\text{N}-\text{N}=\text{C}-\text{S}----\text{S}-\text{Me} \\
\text{Me}
\end{array}
\quad
\begin{array}{c}
\text{HgI Me} \\
|\quad| \\
|\quad| \\
\text{Me I}\quad\text{I}
\end{array}
$$
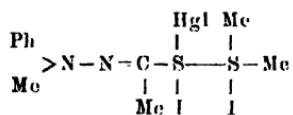
m.p. 127°, gave the following results on analysis :—

0·1867 gave 0·0449 Hg, 0·1527 AgI and 0·0870 BaSO$_4$; Hg = 25·96; I = 48·93; S = 6·40.

0·1070 gave 0·0590 CO$_2$ and 0·0734 H$_2$O; C = 15·06; H = 1·39.

0·1227 gave 4·2 c. c. N$_2$ at 35° and 760 mm.; N = 3·73.

C$_{11}$H$_{17}$N$_2$S$_2$HgI$_3$, requires, Hg = 24·33; I = 46·35; S = 7·79 C = 16·06; H = 2·87.

The above mercaptide nitrite and ethyl iodide; formation of the compound :

$$
\begin{array}{c}
\text{Et} \\
>\text{N}-\text{N}=\text{C}-\text{S}-\text{S}-\text{Et} \\
\text{Et}
\end{array}
\quad
\begin{array}{c}
\text{HgI Et} \\
|\quad| \\
|\quad| \\
\text{Et I}\quad\text{I}
\end{array}
$$
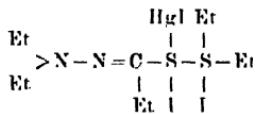
In this case also, the soluble portion (in acetone) was purified by repeated crystallisation. The pure compound was a dull yellow crystalline powder and melted sharply at 73-74°c.

0·2316 gave 0·0572 Hg, 0·1960 AgI and 0·0972 BaSO$_4$; Hg = 24·69; I = 15·74; S = 5·77.*

0·2564 gave 0·0611 Hg; Hg = 23·83.

0·1378 gave 0·0703 CO$_2$ and 0·0296 H$_2$O; C = 15·41; H = 2·58.

0·2470 gave 7·8 C.C. N$_2$ at 31° and 760 mm; N = 3·50.

C$_{11}$H$_{25}$N$_2$S$_2$HgI$_3$ requires, Hg = 24·09; I = 45·9; S = 7·71 C = 15·9; H = 3·07; N = 3·57.

The above three preparations were repeated at least half a dozen times, and in each case, sharply defined compounds of the above composition were obtained.

Phenyl mercaptide nitrite and æthyliodide : formation of the compound, [Ph.MeS$_2$, HgI$_2$, MeI.]

The pure product was obtained by repeated crystallisation and was a dull yellow crystalline powder, m.p. 90°.

0·4616 gave 0·1252 Hg and 0·4353 AgI; Hg = 27·12; I=50·96.

0·1131 gave 0·0551 CO_2; C = 13·29.

$C_8H_{11}S_2HgI_3$ requires, Hg = 26·6; I = 50·66; C = 12·77.

Phenyl mercaptide nitrite and ethyl iodide: formation of the compound [$PhEtS_2$, HgI_2, EtI.]

The soluble portion (in acetone) in this case, first of all appeared as a greenish yellow oil, which soon crystallised on stirring and had the m.p. 59°.

0·2994 gavo 0·0774 Hg and 0·2673 Ag I; Hg=25·85; I=48·24.

0·1302 gave 0·0740 CO_2 and 0·0282 H_2O; C = 15·5; H=1·86.

$C_{10}H_{15}S_2HgI_3$ requires, Hg = 25·64; I=48·84; C=15·39; H = 1·92.

Ph_2S_3Hg and methyl iodide: formation of the compound:

$$\begin{array}{ccc} \text{HgI} & \text{Ph} & \text{Ph} \\ | & | & | \\ \text{Me}-\text{S}-\text{S}-\text{S}-\text{Me} \\ | & | & | \\ | & | & | \end{array}$$
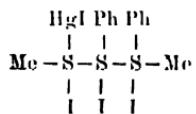
The product was an oil, which on stirring crystallised *en masse*. It was washed several times with acetone when the colour turned to yellowish white. It was insoluble in acetone and melted sharply at 111-112°.

0·3150 gave 0·0615 Hg, 0·2915 AgI and 0·2007 $BaSO_4$; Hg=19·53; I = 50·00; S = 8·75.

0·1410 gave 0·0902 CO_2 and 0·0390 H_2O. C=17·41; H=3·07

$C_{14}H_{16}S_3HgI_4$ requires, Hg = 20·24; I = 51·42; S = 9·71; C = 17·00; H = 1·62.

It will be noticed that whenever a sulphonium compound contains phenyl groups, it becomes insoluble in acetone.

Mercaptide nitrite of thiocarbanilide and ethyl iodid : formation of the compound:

$$\begin{array}{cc} \text{Et} & \text{Et} \\ | & | \\ \text{I}-\text{C}-\text{S}-\text{HgI} \\ | & | \\ \text{Et} & \text{I} \end{array}$$
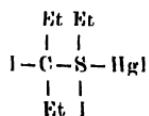
In this case a deep purple-coloured needle-shaped crystalline product was obtained. It was soluble in acetone.

0·3083 gave 0·0890 Hg, 0·2920 AgI and 0·1042 BaSO$_4$;
Hg$=28·73$; I$=51.18$; S$=4.64$.

0·1214 gave 0·0552 CO$_2$ and 0·0232 H$_2$O; C$=12·04$; H$=2·12$.

C$_7$H$_{15}$SHgI$_3$ requires; Hg$=28·09$; I$=53·51$; S$=4·49$; C$=11·8$; H$=2·11$.

Mercaptide nitrite of phenyl imido-phenylcarbamicthiomethylester and Methyl iodide: formation of the compound [*Me$_2$S$_2$, HgI$_2$, MeI.*]

The mercaptide-nitrite was refluxed with methyl iodide. Next day a yellow crystalline mass was found at the bottom of the liquid. The mother liquor was drained off and evaporated on the water bath when a treacle-like mass remained. The yellow crystalline product was exhausted several times with acetone and an insoluble pale yellow crystalline residue melting at 176—180° remained. The treacle-like mass on stirring with a little acetone crystallised *en masse*. The acetone filtrate from both was concentrated to a small bulk and treated with ether when a mealy, pale yellow crystalline mass was precipitated. But, as this had no sharp melting point, it was subjected to fractional crystallisation. The fractions had melting points varying between (147—155°), (153—164°) and (160—176°). The first two were once more dissolved in acetone, filtered off the insoluble portion and again precipitated by ether. The process was repeated five or six times when at last a fairly good crop was obtained, which melted sharply at 160—162°C.

0·3290 gave 0·0791 Hg and 0·3281 AgI; Hg$=29·51$; I$=53·89$.

0·1150 gave 0·0221 CO$_2$ and 0·0218 H$_2$O; C$=5·24$; H$=2·11$.

C$_3$H$_9$HgI$_3$S$_2$ requires Hg$=28·98$; I$=55·21$; C$=5·22$; H$=1·30$.

From its appearance, melting point and analysis, it is thus found to be [Me$_2$S$_2$, HgI$_2$, MeI].

Mercaptide nitrite of phenylimidophenylcarbamicthioethyl ester and ethyl iodide: formation of [*Et$_2$S$_2$, HgI$_2$, EtI.*]

The procedure was almost the same as in the previous instance. On concentrating the acetone solution a portion crystallised out, which had no sharp m.p. (*viz.*, 100-138°).

It was purified by repeated fractional crystallisation; when absolutely pure, it melted sharply at 110-111°.

0·3127 gave 0·0862 Hg and 0·0978 AgI; Hg$=27·57$; I$=51·46$

0·2740 gave 0·0758 Hg; Hg$=27·66$

0·0861 gave 0·0327 CO_2 and 0·0255 H_2O; C=10·37; H=3·29

$C_9H_{15}S_2HgI_3$ requires, Hg=27·32; I=52·05; C=10·37; H=3·29

Mercaptide nitrite from the p. chlorophenylthiourea derivative, viz.,

$$O_2NHgS-<\;\exists\;>-NH-C-S-Hg$$

with $HgNO_2$ above, $\underset{NH}{\overset{}{\|}}$ and $\overset{V}{O}$ below
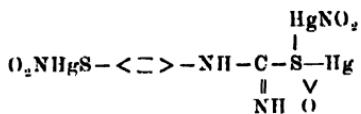
and ethyl iodide : formation of the compounds :

$(SHgI)_2$ and $[Et_2S_2, HgI_2, EtI]$ which agreed with the analysis already published (l.c.p. 109). The di-mercuri-di-iodo-disulphide had reversible phototropic properties.

Phenyl dithiobiazoline disulphide, mercuric iodide and ethyl iodide : synthesis of the compound :

$$
\begin{array}{ccccc}
& & Et\,I & & \\
& & \vee & & \\
PhN-N & HgI & Et & N-NPh \\
\| & | & | & \| \\
C-S & & S-C \\
| & | & | & | \\
SC-S & I & I & S-CS
\end{array}
$$
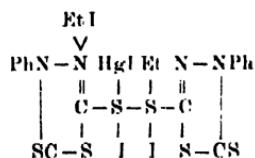
0·3127 gave 0·0512 Hg, 0·2366 AgI and 0·3026 $BaSO_4$; Hg=16·38; I=10·89 and S=13·29*

0·1574 gave 0·1066 CO_2; C=18·47.

Calc. Hg=16·45; I=41·75; S=15·79; C=19·74.

SUMMARY AND CONCLUSION—ENUNCIATION OF LAWS.

From the present investigation it is found (1) that mercuric nitrite acting upon the aryl substituted thiocarbamides brings about their tautomerisation and the divalent sulphur atom thereby acquires additional energy and the capacity to take up the radicals of $Hg(NO_2)_2$, becoming tetravalent, thus :—

$$
\begin{array}{ccccccc}
NH_2 & & NH & & NH & HgNO_2 & NH & HgNO_2 \\
| & Hg(NO_2)_2 & \| & Hg(NO_2)_2 & \| & | & \| & | \\
C=S & \longrightarrow & C-SHgNO_2 & \longrightarrow & C & S-HgNO_2 \to C & S-Hg \\
| & & | & & | & | & | & V \\
NHPh & & NHPh & & NHPh & NO_2 & NHPh & O
\end{array}
$$

---
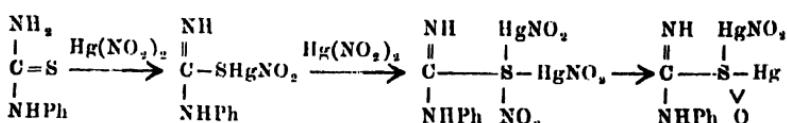
* Owing to the tedious process involved, the sulphur and iodine sometimes came out too low (*vide* "Trans. Chem. Soc." 1916, *109*, p. 135).

(2) That one and the same atom of sulphur cannot retain the load of one $Hg(NO_2)_2$ and an additional $(-HgNO_2)$ groups, but by parting with N_2O_3 becomes an oxynitrite

$$
\begin{array}{c}
\text{HgNO}_2 \\
| \\
-\text{S}-\text{Hg} \\
\vee \\
\text{O}
\end{array}
$$
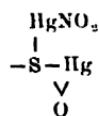
(3) that sometimes, however, mercuric nitrite acts in a twofold character. Not only does it bring about tautomeric changes as indicated above but an atom of hydrogen in the imino-group is simultaneously replaced by its equivalent of (hg); in such cases, the molecule being already loaded with $(-HgNO_2)$ and (hg), the sulphur atom can no longer become tetravalent by taking up a fresh load of $Hg(NO_2)_2$ (Cf. derivatives of diphenyl-thiocarbamide, thiosemicarbazide, etc., p. 6). If, however, there be two mercaptanic sulphur atoms in the molecule, one real and the other potential, then each of these sulphur atoms can individually take up its own quota of $HgNO_2$; *e.g.*,

$$
\begin{array}{c}
\text{SHgNO}_2 \\
| \\
\text{PhNH}\cdot\text{N:C} \\
| \\
\text{S}-\text{HgNO}_2 \\
\wedge \\
\text{Hg}-\text{O}
\end{array}
$$
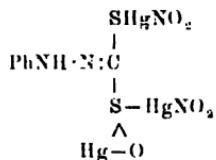
(4) that a mercaptan, however complex its structure might be—provided it is a real mercaptan—does not undergo disruption but has simply the H of SH replaced by $(-HgNO_2)$. If, however, the compound be a potential mercaptan, even if its molecule be of a simple character, *e.g.*, thiocarbamide, it either undergoes disruption or its sulphur atom in view of the superadded energy becomes tetravalent, *i.e.*, $-SHgNO_2$ becomes

$$
\begin{array}{c}
-\text{S}-\text{HgNO}_2 \\
\wedge \\
\text{Hg}-\text{O}
\end{array}
$$
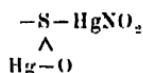
(5) that mercuric nitrite can be used as reagent for the diagnosis of real and potential mercaptans, the *experimentum crucis* being furnished by salts of phenyl sulphocarbazinic acid, which is at once a real and a potential mercaptan and as such the hydrogen attached to the mercaptanic S is simply replaced by $(-HgNO_2)$, whereas, the S of the potential mercaptan possessing additional energy passes on to the tetravalent form, *viz.*,

$$
\begin{array}{c}
-\text{S}-\text{HgNO}_2 \\
\wedge \\
\text{Hg}-\text{O}
\end{array}
$$
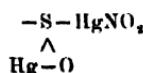
(6) that so great is the affinity of S for the group $(-HgNO_2)$, that it pushes the alkyl radical attached to it to the adjacent nitrogen atom and thus also brings about migration of alkyl radicals.

(7) that the sulphur atoms forming the ring does not possess the property of becoming tetravalent by adding on the radicals of $Hg(NO_2)_2$.

Interaction of Mercuric, Platinic and Cupric Chloride respectively with the Mercaptans and Potential Mercaptans.

BY

PRAFULLA CHANDRA RÂY.

INTRODUCTORY

Some four years ago, the author ventured to put forth the view that the compounds which mercuric, platinic and cupric chloride respectively yield with thiocarbamide and thioacetamide, etc., are of the same nature as those obtained by the interaction of these halides with real mercaptans; *i.e.*, they are chloromercaptides (Proc. Chem. Soc., 1914, *30*, 304). Since then extended investigations have been carried on in this field especially through the aid of mercuric nitrite, (T., 1917, *111*, 101), a reagent which brings about ready tautomeric changes. Iodine also has been found to act similarly on these bodies (Trans. Chem. Soc., 1916, *109*, 698), inasmuch as they behave like iminomercaptans.

It seemed desirable at the outset to undertake a systematic study of the behaviour of mercuric, cupric and platinic chlorides towards some of the typical mono-, and di-mercaptans, namely, ethyl and ethylene mercaptans, phenyldithiobiazoline sulphydrate and thiobiazol disulphydrate.

Ethyl mercaptan has been found to yield under normal conditions a chloromercaptide of the formula $\underset{Et-S}{\overset{Et-S}{>}}Pt-Cl$ and under exceptional conditions a plataso-mercaptide $\underset{Et-S}{\overset{Et-S}{>}}Pt$. Ethylene mercaptan (thioglycol) also yields a chloro-mercaptide of the formula $C_2H_4<\overset{S}{\underset{S}{}}>Pt<\overset{S-C_2H_4}{\underset{Cl\ \ SH}{|}}$. In the formation of this remarkable compound, two molecules of thioglycol simultaneously take part in the reaction; one atom of chlorine of the platinic chloride remains, however, intact.

Phenyldithiobiazoline sulphydrate or rather its potassium salt has been found to be equally reactive towards platinic chloride and it readily yields mercaptides in which platinum functions both as a triad and a diad element as in the case of the simple ethyl mercaptan.

The valency of platinum calls for a few remarks. The salts of the type MX_3 are scarcely represented among the platinum compounds. In the formula $\begin{smallmatrix} Et-S \\ Et-S \end{smallmatrix}>Pt-Cl$, platinum, no doubt, may be made to appear as tetravalent by doubling it. Platinum has its proper place in the eighth or transitional group in the periodic system, which also includes iron and iridium. Now, the chlorides of these metals are represented by the simple formula MCl_3 and not by M_2Cl_6. Moreover, platinum with high atomic weight of 194 will have scarcely two atoms coalesced into the complex $\equiv Pt-Pt \equiv$. Salts containing two atoms of platinum in the molecule have, it is true, been described, but they are of a very complex character.*

The product of the reaction of mercuric chloride with thiocarbamide has been assigned the formula $HgCl_2CSN_2H_4$ (Rosenheim and Meyer, *Zeit., Anorg. Chem.*, 1903, *34*, 62; Ibid, 1906, *49*, 13). In other words, it has been regarded as a mere additive or molecular compound. From considerations based upon analogy the author suggested that it should be formulated as $\left(\begin{smallmatrix} NH_2-C-S-HgCl \\ \| \\ NH \end{smallmatrix} \right)HCl$; *i.e.*, it is the product of the reaction of mercuric chloride upon ψ thiocarbamide (imino-mercaptan). The hydrogen chloride, which is formed, is not liberated in the free state but is simply fixed by the basic complex. It is well-known that formamidine disulphide exists as a salt of a diacid base because of the presence of the amino- and imino- radicals. Convincing proof has recently been obtained in support of the above formula. The compound in question has been found to be sufficiently soluble in water to admit of conductivity measurements. If it were an additive one, in solution it would simply break up into its components. Now, thiocarbamide is practically a non-electrolyte and mercuric chloride is very feebly ionised in solution (Cf. Ráy & Dhar, T., 1913, *103*, 8). The aqueous solution, however, shows an enormous increase in the dissociation, which is of the same order as that of ammonium chloride, proving that the salt is the hydrochloride of a base.

---

* Cossa, Ber. *23* (1890), 2503, Cleve, Bull Soc. Chem. *17* (1872), 289.

Thiocarbamide may be expected to behave in like manner towards platinic chloride and yield a compound of the same type ; thus :—

$$
\begin{array}{ccccc}
 & & NH & & \\
H_2N-CS-NH_2 & PtCl_4 & NH_2-C-S \cdot & & Cl \\
 & & & & \quad \mid, HCl \\
H_2N-CS-NH_2 & & NH & Pt{\scriptstyle<} & \\
 & & \overset{\shortparallel}{NH_2-C-S}{\scriptstyle\diagup} & & Cl \\
\end{array}
$$
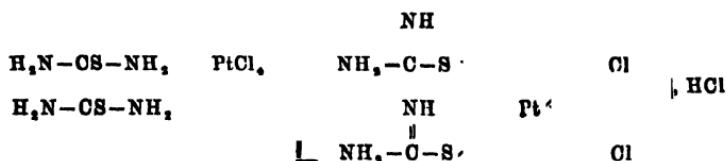
Reynolds and Prätorius-Seidler (Ber., *17*, 307), however, make the compound an additive one of thiocarbamide with platinous chloride and assign to it the formula $2(CSN_2H_4)PtCl_2,HCl$ or $2(CSN_2H_4)PtHCl_3$. That this view of the constitution of the compound is erroneous is evident from another consideration. The platinic chloride on being reduced to the platinous state would set free chlorine, which would, in turn, act upon a portion of the thiocarbamide and foramidine di-sulphide hydrochloride would be simultaneously formed;[*] the derivative in question is, however, free from it.

Cupric chloride is analogous in its behaviour towards thiocarbamide :—

$$
CS{\scriptstyle<}{NH_2 \atop NH_2} \quad \xrightarrow{CuCl_2} \quad \underset{NH}{NH_2-C-S-CuCl}
$$
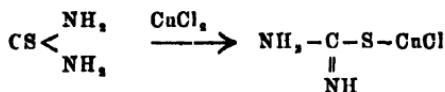
In this case, however, the hydrogen chloride is liberated in the free state and has actually been detected in the filtrate. This is what might have been expected. The chlorides of mercury, platinum and cobalt have the capacity to form complex compounds with ammonia, amines and amino-groups, which, on account of their basic character, can fix a molecule of hydrogen chloride ; while copper being a more positive metal does not possess this property.[†]

The view taken above offers also a ready interpretation of the reaction between platinic chloride and thioacetamide, first studied by Kernakow (Journ. Rus. Chem. Soc., *25*, 613). The Russian chemist formulates the compound as $PtCl_2, 4C_2H_5NS, PtCl_4$ i.e., an additive one of four molecules of thioacetamide with one of

---

[*] It has been actually found that sulphuryl dichloride, which acts as a mild chlorinating agent, behaves exactly like iodine towards thiocarbamide and yields the hydrochloride of the base.

[†] The ammonical compounds of copper are quite distinct from those of platinum and cobalt ; the latter come under Werner's supplementary valencies.

platinous chloride and another of plantinic chloride. This formula, when halved, stands as $PtCl_3$, $2C_2H_5NS$, and is in reality

$$\left[\begin{array}{c} \overset{NH}{\overset{\|}{CH_3-C-S}} \\ CH_3-C-S \\ \overset{\|}{NH} \end{array} >Pt< \begin{array}{c} Cl \\ Cl \end{array}\right] HCl,* $$
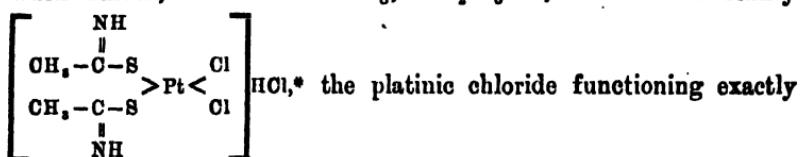
the platinic chloride functioning exactly like mercuric chloride. Thioacetamide here also simply undergoes tautomeric change. One strong argument in favour of the suggested constitution is afforded by the fact that, if platinic chloride were actually to be reduced to the platinous form, the available atoms of chlorine would act upon two other molecules of thioacetamide. It has already been shown that iodine, acting upon thioacetamide, completely breaks it up, the whole of the sulphur being precipitated. (T., 1916, *109*, 698). The product, in question, when treated with carbon bisulphide, however, did not give any sulphur. The corresponding cupric chloride derivative has been found to conform to the formula $\overset{CH_3-C-S-CuCl}{\underset{\overset{\|}{NH}}{}}$; it is, in fact, a chloromercaptide.

Attention may be drawn here to one striking contrast in the deportment of mercuric nitrite and chloride respectively towards these potential mercaptans ; both equally bring about tautomerisation and the initial products are $R-S-HgN_{2}$ and $R-S-HgCl$ respectively ; but the former on account of its readiness to take up an additional molecule of $Hg(NO_2)_2$ undergoes a break-up and assumes the form of a sulphoxy nitrite $\overset{-S-HgNO_2}{\underset{Hg-O}{\wedge}}$, the sulphur atom becoming tetravalent. (See above p. 27.)

The reaction between platinic chloride and thiosemicarbazide is of special interest. The compound obtained has the empirical formula $PtCl_2S_3N_6H_4$. Here also a tautomeric change of the parent thio-body, under the influence of the halide, evidently takes place, resulting in the formation of azomethylene-thiol with the elemination of a molecule of ammonia thus :—

$$\overset{NH_2-C=S}{\underset{NH NH_2}{|}} \rightarrow \overset{NH=C-SH}{\underset{\overline{NH NH_2}}{|}} \rightarrow \overset{NH-C-SH}{\underset{N}{\diagdown\|}}$$

$$\overset{PtCl_4}{\longrightarrow} \left[(NH-C-S)_2 \equiv Pt-Cl\right]HCl$$

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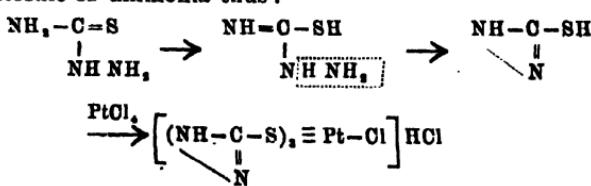
* In this formula there is a deficiency of only one atom of hydrogen.

The mercaptan now acts upon platinic chloride in the usual manner, the resulting product being a chloromercaptide. Of the three molecules of hydrogen chloride generated, one is fixed by the complex body because of its basic character due to the presence of several imino-residues.

EXPERIMENTAL.

I.

MERCAPTANS AND METALLIC HALIDES.

(a) Ethyl mercaptan and platinic chloride.

Hofmann and Rabe (Zeit. Anorg. Chem., 1897, *14*, 294) maintain that if to chloroplatinic acid be added ethyl mercaptan in alcoholic solution, at first an orange yellow platinimercaptide $Pt(SEt)_4$ is formed which, when heated to $100°$ in a vacuum, readily yields plataso-mercaptide. It has been found, however, that by the interaction of platinic chloride and ethyl mercaptan, a platini-chloro-mercaptide of the formula $\begin{matrix} Et-S \\ Et-S \end{matrix}>Pt-Cl$ is almost invariably formed and that it is only under exceptional conditions that a plataso-mercaptide $Pt(SEt)_2$ can be obtained. If the components be added in a half-hazard manner, a mixture of the two compounds in varying proportions is formed. The *modus operandi* is therefore given here somewhat in detail. A concentrated solution of chloro-platinic acid in alcohol is taken and a very dilute solution of ethyl mercaptan in alcohol is added to it from time to time under vigorous shaking; for the first few moments, only a deep orange coloration is noticed, but shortly after a precipitate begins to put in an appearance. Care should be taken that the platinic chloride be always in sufficient excess. The orange yellow precipitate is allowed to settle for half an hour and sucked on the filter pump and washed preferably with alcohol and dried in a vacuum. If the conditions of reaction be reversed, *i.e.*, if a dilute solution of platinic chloride in small instalments be added to an excess of a concentrated solution of ethyl mercaptan, under vigorous shaking, for a few minutes the solution remains clear, after which a yellow salt begins to separate out, which is very nearly pure platasomercaptide $Pt(SEt)_2$.

Result of Analysis.

Prep. 1.   0·050 gave 0·0276 Pt, 0·022 AgCl and 0·065 $BaSO_4$.*

         Pt=55·20 ; Cl=11·33 ; S=17·85.

Prep. 2.   0·0502 gave 0·0283 Pt and 0·0375 $BaSO_4$.

         Pt=56·37 ; S=18·47.

Prep. 3.   0·0226 gave 0·0126 Pt. ∴ Pt=55·75.

Prep. 4.   0·0258 gave 0·0145 Pt. ∴ Pt=56·2.

	Found				Calc. for $(EtS)_2 PtCl$.
	1	2	3	4	
Pt	55·2	56·37	55·75	56·2	55·02
Cl	11·33	...	...	...	10·09
S	17·85	18·47	...	...	18·19

Plataso-mercaptide $Pt(SEt)_2$.

0·1317 gave 0·0814 Pt, 0·012 AgCl and 0·1985 $BaSO_4$.

	Found	Calc.
Pt	61·81	61·43
Cl	2·25	...
S	20·71	20·24

Contamination with distinct traces of the previous compound, which is invariably formed, accounts for the presence of chlorine.

(*b*) *Phenyl-dithiobiazoline sulphydrate and platinic chloride.*

The potassium salt of the mercaptan was used. To a concentrated aqueous solution of it was added under stirring a dilute solution of chloro-platinic acid in a thin stream. A yellow precipitate came down, which on drying became orange. Here also, as in the case of ethyl mercaptan, the platinum atom functions as trivalent and divalent.

Result of Analysis.

0·980 gave 0·0213 Pt and 0·1805 $BaSO_4$.

	Found	Calc. for $\left(\begin{array}{c} PhN-N \\ \| \quad CS \\ SC-S \end{array} \right)_2 Pt$	Calc. for $\left(\begin{array}{c} Ph-N-N \\ \| \quad CS \\ SC-S \end{array} \right)_2 Pt$	Mean.
Pt	26·63	22·33	30·12	26·23
S	31·00	33·14	29·18	31·16

Chlorine was found to be absent.

---

* The estimation throughout was effected by fusion with sodium nitrate and sodium carbonate.

A second preparation gave Pt = 25·32.

$$\begin{bmatrix} Ph-N-N \\ \overset{\|}{\underset{|}{C}}S \\ SO-S \end{bmatrix}_4 Pt \quad \text{requires, } Pt = 17\cdot73\,; \ S = 35\cdot10.$$
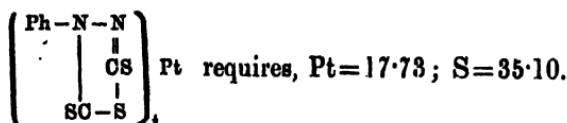
It will thus be seen that here also the compound corresponding to the tetravalent platinum is never formed but a mixture of almost equal proportions of that yielded by trivalent and divalent platinum. As has been previously shown it is only in the case of the simple mercaptans like ethyl mercaptan that the products can be isolated in a pure state by taking proper precautions.

In the above reactions the free HCl of H_2PtCl_6 acting upon a portion of the potassium mercaptide, liberates the corresponding mercaptan, which is insoluble in water and the available chlorine atom of the tetravalent platinum chloride also acting upon potassium mercaptide gives rise to the disulphide equally insoluble. The dried precipitate was therefore exhausted first with alcohol and afterwards with benzene for the extraction of the impurities. The colour of the compound, which was yellow, after this treatment, became orange.

(c) *Phenyl-dithiobiazoline sulphydrate and cupric chloride.*

The potassium salt in aqueous solution was used as it was found to be more reactive than the mercaptan itself.

To a concentrated solution was added a dilute solution of cupric chloride with vigorous stirring. A gelatinuous precipitate was obtained which, when dried in a vacuum desiccator, was of a yellow colour. It was cupric mercaptide.

Result of Analysis.

0·1518 gave 0·247 CuO and 0·4002 $BaSO_4$.

Found	Calc. for $(PhN_2C_2S_3)_2Cu$.
Cu = 13·00	12·37
S = 36·21	37·39

Chlorine was proved to be absent.

By the reverse process, *i.e.* by using a dilute solution of the mercaptide and concentrated solution of cupric chloride, the

corresponding chloro-mercaptide was obtained but it was contaminated with the mercaptide.

	Found	Calc. for $(PhN_2C_2S_3)CuCl$.
$Cu =$	17·63	19·60
$S =$	30·66*	29·63
$Cl =$	6·28	10·96

II

DIMERCAPTANS AND METALLIC HALIDES.

(a) *Ethylene mercaptan (thioylycol) and platinic chloride.*

A dilute alcoholic solution of platinic chloride is gradually added under vigorous stirring to a concentrated alcoholic solution of thioglycol. A yellowish brown precipitate is obtained, which is washed with alcohol and dried in a vacuum.

It conforms to the formula :

$$C_2H_4 \!\!<^S_S\!\!> Pt <^{S-C_2H_4}_{\substack{|\\Cl\ SH}}$$
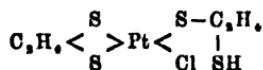
It is evidently a remarkable compound ; two molecules of thioglycol which take part in the reaction have had three hydrogen atoms of the thiol-groups (SH) replaced by the three atoms of chlorine of platinic chloride and only one (SH) group remains intact. In orther words, it is three fourths a mercaptide and one fourth a chloride.

Result of Analysis.

0·0788 gave 0·0373 Pt, 0·032 AgCl and 0·1608 $BaSO_4$.

	Found	Calc.
Pt	47·34	46·96
Cl	10·05	8·58
S	28·03	30·93

* The analysis was conducted by fusing the substance with sodium nitrate and sodium carbonate. Although the "melt" is evaporated with hydrochloric acid, it is not easy to convert the whole of the nitrate into the chloride and hence the p. c. of sulphur often comes out too high.

The compound is by no means pure; there is reason to conclude that a simpler mercaptide of the formula $C_2H_4 {<}^S_S {>} Pt {<}^{Cl}_{Cl}$; $(Pt = 54\cdot39$ $Cl = 19\cdot88$; $S = 17\cdot91)$ is simultaneously formed in small quantities; the higher value of platinum and chlorine and the lower value of sulphur is thus accounted for. Attempts to obtain the chloro-mercaptide in a purer form proved unsuccessful.

(b) *Thiobiazole-disulphydrate and mercuric chloride.*

An alcoholic solution of the di-mercaptan prepared according to the method of Busch (Ber., 27, 2518) was added in a thin stream to an aqueous solution of mercuric chloride under vigorous stirring. A granular faintly yellow precipitate was obtained, which was sucked dry in the pump, washed with alcohol and dried in a vacuum over sulphuric acid. The compound conformed to the formula

$$
\begin{array}{ccc}
\text{N--N} & \text{N--N} & \text{N--N} \\
\text{\textbardbl \ \ \textbardbl} & \text{\textbardbl \ \ \textbardbl} & \text{\textbardbl \ \ \textbardbl} \\
\text{ClHg--SC} & \text{CS--SC} & \text{CS--SC} \quad \text{CS--HgCl} \\
\vee & \vee & \vee \\
\text{S} & \text{S} & \text{S}
\end{array}
$$
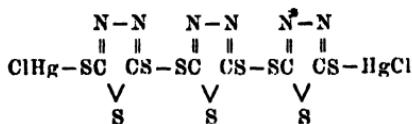
i.e., it is the product of the condensation of three molecules of the di-mercaptan (Cf. similar chain compounds under the action of mercuric nitrite). Although no nitrous acid is present here to bring about oxidation, evidently so great is the tendency towards the formation of the chain compounds of sulphur that aerial oxidation suffices for the purpose.

Result of Analysis.

$0\cdot310$ gave $0\cdot154$ Hg ; $0\cdot1479$ gave $0\cdot328$ BaSO$_4$.
$0\cdot1204$ gave $10\cdot20$ c. c. N$_2$ at $29°$c and 760 m. m. pressure.
$0\cdot1656$ gave $0\cdot0144$ AgCl.

	Found	Calc. for $(C_2N_2S)^3S_6(HgCl)_2$.
Hg	45·29	43·65
S	30·45	31·48
N	8·95	9·18
Cl	6·05	7·76

The compound is seldom obtained pure ; products have sometimes been obtained in which only one molecule of the mercaptan takes part in the reaction ; they are, of course, contaminated with the above and vice-versâ.

	Found	Calc. for $(C_2N_2S)S_2(HgCl)_2$.
Hg	60·02	64·63
S	21·20	15·51
N	5·85	4·52
Cl	9·84	11·47

This preparation evidently contained an admixture of the previous one ; the p.c. of mercury is thus lowered and that of sulphur increased.

(c) *Thiobiazole-disulphydrate and cupric chloride.*

To the concentrated solution of the potassium salt in dilute alcohol was added a dilute solution of cupric chloride. A yellow gelatinous compound was obtained, which was dried in a vacuum desiccator. It conformed to the formula $(N_2C_2S)S_2Cu + 2H_2O$.

Result of Analysis.

0·2362 gave 0·0747 CuO and 0·6854 $BaSO_4$.

	Found	Calc.
Cu	25·26	25·66
S	39·85	38·79

By the reverse process *i.e.*, by using a concentrated solution of cupric chloride and a dilute solution of the mercaptide, an impure chloro-mercaptide was obtained.

	Found	Calc. for the corresponding chloro-mercaptide.
Cu	32·79	36·70
Cl	4·74	20·52

It will thus be seen that the tendency is always towards the formation of the mercaptide. As the series of mercaptides and chloro-mercaptides described above are insoluble in ordinary solvents, it is difficult to purify them.

III.

POTENTIAL MERCAPTANS AND METALLIC HALIDES.

(a) *Thiocarbamide and mercuric chloride.*

. The components were acted upon both in alcoholic and in aqueous solution ; a copious white precipitate was obtained which was washed with alcohol and dried in a vacuum. The filtrate was found to be neutral.

Result of Analysis.

0·2200 gave 0·1445 HgS; 0·1980 gave 0·1637 AgCl and 0·1446 BaSO$_4$.
0·1456 gave 10·4 c.c. N$_2$ at 22°c and at 760 m. m. pressure.
0·1313 gave 0·184 CO$_2$ and 0·0192 H$_2$O.

	Found	Calc. for CSN$_2$H$_4$HgCl$_2$.
Hg	56·62	57·64
Cl	20·45	20·46
S	10·03	9·22
N	8·14	8·07
C	3·82	3·46
H	1·63	1·15

CONDUCTIVITY MEASUREMENT.

V	T	μ
256	21°·5	98·96

There were evidently two ions in solution and the dissociation is of the order of ammonium chloride.

(*b*) *Thiocarbamide and platinic chloride.*

An acetone solution of thiocarbamide was cautiously added under constant stirring to platinic chloride solution. An orange yellow precipitate was obtained which was washed with alcohol and dried in a vacuum.

Prep. 1.   0·0838 gave 0·0365 Pt, 0·0910 BaSO$_4$ and 0·0804 AgCl.
       0·0582 gave 6·6 c. c. N$_2$ at 33°c and at 760 m. m. pressure.
Prep. 2.   0·0196 gave 0·0540 BaSO$_4$ and 0·0482 AgCl.

	Found		Calc. for C$_2$N$_4$H$_7$S$_2$Pt.Cl$_3$.
	1	2	
Pt	43·56	——	43·01
S	14·91	14·95	14·17
Cl	23·74	24·03	23·57
N	12·39	——	12·40

(c) *Thiocarbamide and cupric chloride.*

To an alcoholic solution of cupric chloride was added under constant stirring an alcoholic solution of the carbamide. The granular precipitate when dried in a vacuum had a white colour with a faintly bluish tint.

Result of Analysis.

0·0608 gave 0·9275 CuO.

0·0443 gave 6·2 c.c. N_2 at 31°c and 760 m. m. pressure.

9·0588 gave 0·0483 AgCl and 0·0780 $BaSO_4$.

Found		Calc. for NH=C—S—CuCl. NH₂
Cu	36·09	36·38
S	18·23	18·43
Cl	20·31	20·44
N	15·41	16·12

(*a*) *Thioacetamide and platinic chloride.*

Thioacetamide in alcoholic solution was added under stirring to platinic chloride solution and the precipitate which was brownish yellow, was treated as before.

Result of Analysis.

Prep. 1. 0·0524 gave 0·0222 Pt.

0·0856 gave 0·0782 AgCl.

Prep. 2. 0·0852 gave 0·0360 Pt and 0·0785 AgCl.

Prep. 3. 0·0334 gave 0·0144 Pt.

0·0813 gave 5·2 c.c. N_2 at 32°c and at 760 m. m. pressure.

Prep. 4. 0·1457 gave 0·0630 Pt; 0·1697 $BaSO_4$ and 0·1313 AgCl.

	Found				Calc. for $C_4H_9N_2S_2PtCl_3$.
	1	2	3	4	
Pt	42·40	42·26	43·11	43·83	43·20
S	——	——	——	15·99	14·23
Cl	22·59	22·79	——	22·29	23·68
N	——	——	7·01	——	6·23

(*b*) *Thioacetamide and cupric chloride.*

The method of preparation was exactly the same as in the preceding case ; the colour of the salt was almost white. The alcoholic filtrate when distilled off was found to contain free hydrogen chloride.

Result of Analysis.

Prep. 1. 0·0555 gave 0·0250 CuO.

0·0595 gave 4·6 c.c. N_2 at 29·5°c and 760 m. m. pressure.

Prep. 2. 0·2818 gave 0·1261 CuO, 0·42 $BaSO_4$ and 0·2250 AgCl.

	Found		Calc. for $CH_3-C-S-CuCl.$ $\overset{\parallel}{NH}$
	1		
Cu	35·95	35·78	36·59
S		20·46	18·53
Cl		19·75	20·55
N	8·57		8·11

There can scarcely be any question as regards the purity of the compound analysed. The unusually high percentage of sulphur calls for an explanation. It was treated with CS_2, but no free sulphur could be detected. It is just possible that the free hydrogen chloride liberated, acts upon another portion of the thioacetamide and the latter undergoes the usual decomposition : H_2S and free sulphur being formed. The $H_{11}S$ in turn would act upon the copper chloride forming copper sulphide and the free sulphur is probably adsorbed. This anomaly was only found in the case of thioacetamide.

(a) Thiosemicarbazide and platinic chloride.

On adding an aqueous solution* of semicarbazide to platinic chloride under varying conditions, dark brown crystalline precipitates were yielded but no definite formula could be assigned to any of these.

Result of Analysis.

	1	2	3	4
Pt	49·68	46·14	50·56	51·07
Cl	11·90	24·65	15·16	19·29
S	15·73	13·27	12·43	11·35
N	—	—	—	12·55

It will thus be seen that each of the above four preparations had a different composition ; evidently, mixtures were invariably obtained. After repeated failures the method as described below gave satisfactory result.

A concentrated aqueous solution of thiosemicarbazide was taken to which was added a dilute solution of platinic chloride. No immediate precipitation took place. In the course of half an hour a granular, hard, pale white crop began to be deposited. The solution was

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* Thiosemicarbazide was found to be almost insoluble in cold alcohol, ether and acetone.

allowed to stand over-night. The granules when collected and dried
had a yellowish white colour.

Result of Analysis.

0·2864 gave 0·1171 Pt, 0·1719 AgCl and 0·387 PaSO$_4$.
0·0381 gave 6·2 c.c. N$_2$ at 30° and 760 m. m. pressure.

	Found	Calc. for PtCl$_2$S$_3$N$_6$H$_4$.
Pt	40·89	40·04
S	18·56	19·78
Cl	14·85	14·63
N	17·97	17·31

Summary and conclusion.

1. While mercuric chloride by interaction with a mercaptan
yields invariably a chloro-mercaptide, thus :

$$RSH + ClHgCl = RSHgCl + HCl$$
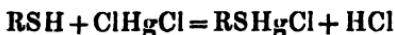
cupric and platinic chlorides, on the other hand, have generally a
tendency towards the formation of mercaptides or at best mercaptides
with only a slight admixture of chloro-mercaptides.

2. Platinum in relation to mono-mercaptanic radicals either
functions as trivalent or divalent.

3. The above chlorides bring about tautomeric changes in
thiocarbamide and thioacetamide. Thiosemicarbazide also with
platinic chloride undergoes similar transformation and yields a
degradation product, which may be regarded as azomethylene-thiol :

NH$_2$—C=S $\longrightarrow$ NH=C—SH $\longrightarrow$ NH—C—SH.
         | |
       NH NH$_2$ NH NH$_2$ N
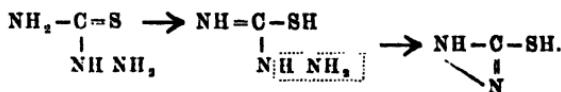
4. The chloro-mercaptides of the potential mercaptans, inasmuch
as they contain one or more imino-groups, have the capacity to fix a
molecule of hydrogen chloride ; in fact, they are hydrochlorides of
complex bases.

WANDERING OF HYDROGEN FROM THE ALPHA POSITION TO THE SULPHUR ATOM IN THIOPHENE AND THIOPHENE CARBOXYLIC ACIDS :

BY

PRAFULLA CHANDRA RÁY.

By the interaction of $\alpha\beta$-thiocrotonic acid and mercuric chloride a compound was obtained to which was assigned the constitution as represented below :—

$$CH_3 \cdot C = C - C = O$$
$$\underset{S - O}{V \quad |}$$
$$|$$
$$HgCl$$
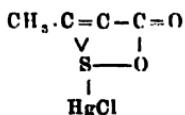
(Trans. Chem. Soc., 1917, *111*, 511)

It was evident that the elimination of an atom of chlorine as hydrochloric acid could only take place by the hydrogen of the carboxyl group shifting itself to the sulphur atom giving rise to a mercaptan and this latter in turn acting upon mercuric chloride in the usual way. The sulphur atom also now becomes tetravalent with the formation of a sulphonium derivative.

It occurred to the author that as $\alpha\beta$-thiocrotonic acid may be regarded as the lowermost homologue of α-thiophene carboxylic acid the latter should also yield with mercuric chloride a corresponding chloromercaptide. :

$$HC - CH$$
$$\underset{HC \quad C - C = O}{\| \quad \|}$$
$$\underset{S - O}{V \quad |}$$
$$|$$
$$HgCl$$
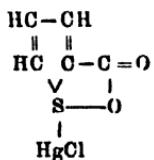
This anticipation has been fulfilled.

The wandering of the hydrogen of a carboxyl group to the nuclear atom in the ortho position is not without many parallels. In the case of orthoaldehydic acids, Liebermann has found that

$$\underset{C \cdot COOH}{C \cdot CHO} \text{ group changes to } \underset{C - C}{\overset{C - CHOH}{\underset{\|}{O}}} > O$$
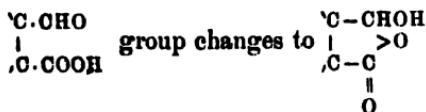
(Ber., *19*, 765, 2288)

Peculiar interest attaches to the formation of the above chloro-mercuri-compound, when it is borne in mind that $\alpha\beta$-thiocrotonic acid is itself the product of a tautomeric change in its parent sub-stance, namely thioacetamide thus :—

$$CH_3 \cdot CS \cdot NH_2 + CH_2Cl \cdot COOH$$

$$= CH_3 \cdot C \cdot S \cdot H \quad ClH_2 \cdot C \cdot COOH$$
$$\underset{NH}{\overset{\|}{}}$$

$$= CH_3 \cdot C = C \cdot COOH + NH_4Cl$$
$$\overset{V}{\underset{S}{}}$$
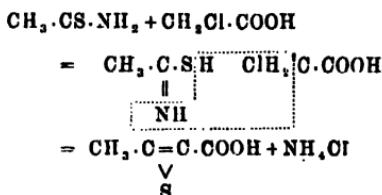
Volhard, however, has shewn that thiophene itself with mercuric chloride gives two mercuri chlorides of the formula C_4H_3SHgCl and $C_4H_2S (HgCl)_2$ respectively. He does not express any opinion as regards their constitution but simply contents himself by saying that it is the hydrogen in the α-position that is successively replaced by the chlorine of corrosive sublimate with the formation of hydro-chloric acid (Ann. *267*, 172).

From the context one would gather that the HgCl groups remain attached to the two neighbouring carbon atoms.

The mechanism of the reaction in the present instance is easily intelligible on the hypothesis that thiophene acts as a potential mercaptan in the presence of suitable reagents, e.g. $HgCl_2$, thus

$$\begin{array}{ccc} HC-CH & & HC-CH \\ \| \quad \| & \xrightarrow{HgCl_2} & \| \quad \| \\ HC \quad CH & & HC \quad C \\ \overset{V}{S} & & \overset{\diagup\!\!\diagup}{S} \\ & & \quad | \\ & & HgCl \end{array}$$
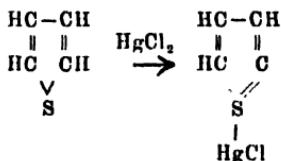
the divalent sulphur atom becoming tetravalent.

When, however, a second molecule of mercuric chloride acts upon the above, a compound of this formula is momentarily formed.

$$\begin{array}{ccc} HC-CH & & HC-CH \\ \| \quad \| & \longrightarrow & \| \quad \| \\ HC \quad C & & C \quad C \\ \diagdown\!\!\diagup & & \diagdown\!\!\diagup \\ Cl-S-HgCl & & ClHg-S-HgCl \\ | & & \\ HgCl & & \end{array}$$
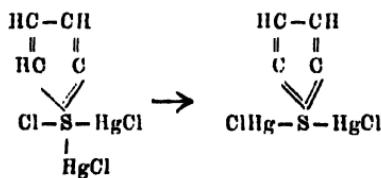
i.e., the tetravalent sulphur atom becomes hexavalent, but as this configuration is unstable, a molecule of hydrogen chloride goes off and a dimercuri-chloride is generated. It is equally possible that by the further action of $HgCl_2$ the tetravalent sulphur atom becomes

hexavalent by the wandering of the second labile dydrogen in the a-position giving rise to a mercaptan and consequently to a chloromercaptide. The author has recently succeeded in preparing a compound, which has the empirical formula

$$[Et_2S_2, HgI_2, EtI]HgI_2*$$
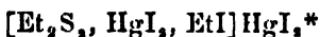
i.e., cthyldisulphonium-mercuri-iodide (Trans. Chem. Soc., 1916, *109*, 135) in combination with an additional molecule of HgI_2. It may be represented by the graphic formula
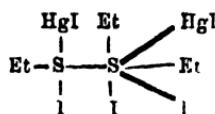
In other words, one atom of sulphur in the chain behaves as tetravalent and the other as hexavalent.

Volhard also noticed that a-methyl thiophene gives only one monomercurichloride ; this is readily explained as the adjacent labile hydrogen being already replaced by the CH_3 group is no longer available for combination with the chlorine atom.

One strong argument in favour of the constitution suggested above is based upon the strong affinity of sulphur for the $HgCl$ group, which justifies the very nomenclature of mercaptans. It is scarcely possible that the $HgCl$ radical would be linked to the carbon atom in preference to the sulphur atom. For, in that case the hydrogen in the β-position might equally have been replaced ; moreover, benzene might also have been expected to have one or more of its hydrogen atoms replaced by (HgCl).

EXPERIMENTAL.

Thiophene a-carboxylic acid was prepared according to the method of Paal and Tafel (Ber., *18*, 458). The aqueous solution when treated with a solution of mercuric chloride gave a white gelatinous precipitate, which was dried in a vacuum over sulphuric acid.

Result of analysis :—

0.0625 gave 0.0247 AgCl and 0.038 BaSO$_4$.

0.248 gave 0.1574 HgS.

	Found	Calc. for $C_5H_2O_2SHgCl$
Hg	54.69	55.17
Cl	9.77	9.79
S	8.35	8.83

* Details of this interesting compound will be communicated later on.

SYNTHESIS OF $\alpha\beta$-THIOCROTONIC ACID.

By Prafulla Chandra Rây and Manik Lal Dey.

Monochloroacetic acid and thioacetamide have been found to interact in acetone solution with the production of $\alpha\beta$-*thiocrotonic acid* according to the following equation :

$$CH_3 \cdot CS \cdot NH_2 + CH_2Cl \cdot CO_2H = CH_3 \cdot C \!=\!=\! C \cdot CO_2H + NH_4Cl$$
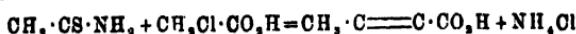
$$\underset{S}{}$$

Evidently the thio-body undergoes a tautomeric change thus :

$$CH_3 \cdot C:S \rightarrow CH_3 \cdot C \cdot SH \quad Cl \cdot H_2 C \cdot CO_2H$$
$$\underset{NH_2}{|} \qquad \underset{NH}{\overset{\|}{|}}$$

Experimental.

Molecular proportions of thioacetamide and monochloroacetic acid are separately dissolved in the minimum quantity of anhydrous acetone, the solution mixed and set aside. In the course of twenty-four hours a precipitate of ammonium chloride is noticed, which increases day by day, and after about a week the ammonium chloride is filtered off.

The filtrate is now freed from acetone by evaporation at the ordinary temperature when a heavy, orange-yellow oil of an unpleasant odour remains with a few crystals of ammonium chloride. The oil is dissolved in ether, the solution filtered and evaporated, and the oil allowed to crystallise in a desiccator.

It takes about three months to obtain a good crop of crystals, but occasionally they appear earlier. The crystals are purified by recrystallisation, first from acetone and then from alcohol. The first crop of crystals obtained in this way were pure, had a faint yellow tint, and melted sharply at 135°; when the ethereal solution was evaporated slowly, transparent, rhombic plates were obtained.

The acid is fairly soluble in water, readily so in alcohol, acetone, or ether, and sparingly so in chloroform. Its solution rapidly decolorises bromine water and alkaline permanganate solution, thus proving its unsaturated character.

The *barium* salt forms shining, rhombic plates, the *lead* salt white, needle-shaped crystals, and the *silver* salt is obtained as a faintly yellow, gelatinous precipitate.

Attempts to analyse the acid met with no success; thus when mixed with concentrated nitric acid in a sealed tube for the purpose of estimating sulphur, it explodes violently. Fusion with sodium carbonate and nitrate is also attended with explosion. The barium, silver and lead salts were therefore analysed.

Barium salt:

0·1238 gave 0·0800 $BaSO_4$.   Ba = 37·99.

0·1118 ,,   0·1460 $BaSO_4$.   S = 17·94.

    $C_8H_6O_4S_2Ba$ requires Ba = 37·23 ; S = 17·44 per cent.

Lead salt:

0·1682 gave 0·1368 CO_2 and 0·0514 H_2O.   C = 22·10; H = 3·40.*

0·0628 ,,   0·0432 $PbSO_4$.   Pb = 46·97.

	Found	Calc. for $C_8H_6O_4S_2Pb$.
Pb	... 46·97	47·17
C	... 22·10	22·05
H*	... 3·40	1·38

Silver salt :

0·1250 gave 0·0610 Ag.   Ag = 48·80.

    $C_4H_3O_2SAg$ requires Ag = 48·43 per cent.

The formula of the compound is thus fully established.

Mercury salt.—With mercuric chloride the acid forms a compound containing an atom of chlorine, to which the constitutional formula

$$CH_3 \cdot C = C \cdot CO$$
$$\begin{array}{cc} \vee & | \\ S & - O \\ | & \\ HgCl & \end{array}$$
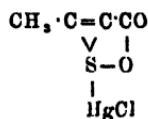
must be assigned.

Here evidently a wandering of the hydrogen of the carboxyl to the sulphur atom takes place, a *mercaptan* being formed.

I.   0·0790 gave 0·0524 HgS.   Hg = 57·17.

II.   0·0828 ,,   0·0553 HgS.   Hg = 57·57.

    0·0920 ,,   0·0620 $BaSO_4$.   S = 9·26.

    0·0812 ,,   0·0339 AgCl.   Cl = 10·24.

$C_4H_3O_2ClSHg$ requires Hg = 57·06; S = 9·13; Cl = 10·13 per cent.

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* Several samples of organic mercury compounds had been previously analysed in the combustion tube and hence traces of mercury vapour were carried into the calcium chloride tube, which increased the apparent percentage of hydrogen.

Action of Hydriodic Acid.

The concentrated aqueous solution of the acid was repeatedly treated with concentrated hydriodic acid and evaporated to dryness on the water-bath so as completely to remove every trace of the free halogen acid.   The residue was dissolved in water, and on treatment with silver nitrate solution it gave a copious, pale yellow precipitate of *silver β-iodo-aβ-thiobutyrate* :

$$CH_2 \cdot CI - CII \cdot CO_2Ag$$
$$\overset{V}{\underset{S}{}}$$
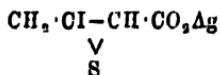
0·0256 gave 0·0174 AgI.*   Ag=31·23 ; I=36·73.

$C_4H_4O_2ISAg$ requires Ag=30·77 ; I=36·18 per cent.

Molecular Conductivity and Dissociation Constant.

The conductivity of the acid and of its barium salt at different dilutions was determined.   The dissociation constant of the acid and the degree of dissociation of the salt were calculated from the data.

	Acid. $t=24°$.			Barium Salt. $t=25·3°$.	
v in litres.	μ (equivalent conductivity).	K.	v	μ (equivalent conductivity).	a
60	42·6	0·000020	200	94	0·75
120	58	0·000019	600	102	0·81
240	77	0·000018	1,200	108	0·86
480	101	0·000017	2,200	117	0·93
960	132	0·000016	4,200	122	0·97
	410 mean	0·000018	10,200	126	1·00

K for crotonic acid $= 0·000002$; as a general rule, the substitution of sulphur for hydrogen in organic acids increases the dissociation constant.   Thus, in the present instance the value of K for the thio-compound is considerably higher.

Interaction of the Potassium Salts of Phenyl-dithio-biazoline-sulphydrate and Thio-biazole-disulphydrate with the Halogenated Organic Compounds.

By

Prafulla Chandra Rây

Prafulla Chandra Guha

and Radha Kishen Das.

Gabriel and, later on, Holmberg treated sodium ethylmercaptide with chloroform and obtained orthotrithioformic-ethylester $CH(SEt)_3$ (Ber., *10*, 185, Ibid *40*, 1740)—a compound which is the sulphur analogue of Williamson and Kay's ether. In the present investigation it will be shown that the potassium atom of the complex cyclic mercaptides named above is singularly reactive towards the halogen atoms of the organic bodies of divers types. Thus, chloroform, bromoform and iodoform yield compounds with the potassium monomercaptide, which may be represented by the general equation given below :—

$$3K\text{-}Biaz + CHX_3 = 3KX + CH(Biaz)_3$$
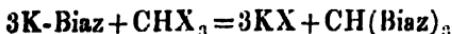
where X represents an halogen atom and "Biaz" the radicle of the complex mercaptide. Nitro-chloroform acts exactly like chloroform but is far more reactive than the latter and the reaction may be expressed by the following equation :—

$$3Biaz\text{-}K + Cl_3CNO_2 = NO_2C(Biaz)_3 + 3KCl.$$

The behaviour of tribromoresorcin, benzal-chloride, monochloro acetic acid and ethylene dibromide has also been found to be of identical nature. The potassium salt of thiobiazole-disulphydrate, on the other hand, acts almost as an inert substance towards chloro-, bromo-, and iodoform.

It is evident that the presence of two (SK) groups of the dimercaptide exercises a sort of inhibitory influence on the halogen atoms. The reactivity of these halogenated bodies can be, however, materially enhanced by further substituting the remaining hydrogen atom

by a nitro-group. For instance, nitro-chloroform very readily acts upon the dimercaptide even in the cold. The introduction of an additional negative group has thus a marked effect. The reaction may be represented as follows :—

$$3\left(KS-\underset{\underset{S}{V}}{\overset{\overset{N-N}{\|\ \ \|}}{C}}\ \ O-SK \right) + 2Cl_3CNO_2 =$$

$$\underset{\underset{\overset{|}{\underset{\overline{\hspace{0.5cm}C\ \ \ \ \ O\ \ \ \ \ C\hspace{0.5cm}}}}{|}}{\overset{N-N}{\|\ \ \|}}}{S-\overset{N-N}{\underset{\overset{V}{S}}{C}}\ \ \overset{}{C-S}}\ \ S-\overset{N-N}{\underset{\overset{V}{S}}{C}}\ \ C-S\ \ S-\overset{N-N}{\underset{\overset{V}{S}}{C}}\ \ C-S \quad +6KCl+N_2O_3$$
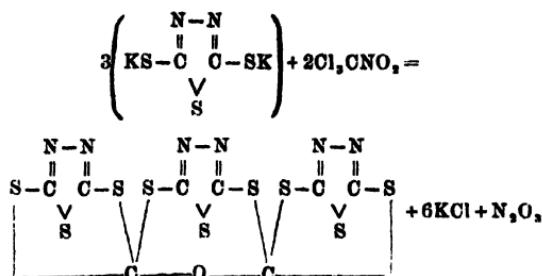
Nitrous fumes (N_2O_3) escape during the reaction and an atom of oxygen as shown above forms the connecting link between the two carbon atoms of the residues of two molecules of nitrochloroform.

Ethylene-dibromide reacts upon the potassium mono-mercaptide, but only one atom of bromine is acted upon by the potassium atom resulting in the formation of

$$BrCH_2-CH_2-S-\underset{\underset{S-CS}{|}}{\overset{\overset{N-NPh}{\|}}{C}}\ \Big|\ .$$
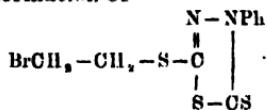
Monochloro acetic acid, benzal-chloride and ethylene dibromide, no doubt, act upon the dimercaptide but the products of the reaction have been found to be insoluble in the ordinary solvents and thus cannot be purified.

EXPERIMENTAL.

K-Biaz* with iodoform, bromoform and chloroform.

The reacting substances were taken in molecular proportions, *i.e.* 3 molecules of the mercaptide were treated with one molecule of the halogenated body in alcoholic solution. The reaction mixture was refluxed on the water bath for several hours. An insoluble mass was obtained consisting of the potassium halide and the organic derivative. The mixture was then triturated in a mortar with water. The aqueous filtrate, on evaporation, gave crystals of the potassium

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* For brevity's sake "Biaz" has been used for (R-S) where R is the complex

$$\text{radicle}\ \ \underset{\underset{SC-S}{|}}{\overset{\overset{PhN-N}{|\ \ \ \ \|}}{}}\ C-$$
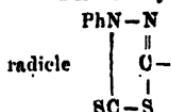
halide. The insoluble portion was dried and dissolved in benzene; on evaporation of the solvent, an oily liquid was obtained. It was redissolved in benzene and precipitated by alcohol as an oil. On keeping, the oil turned into a yellow powder. As it was difficult to get rid of the last trace of iodoform, the powder was repeatedly washed with alcohol and dried in the steam oven, till the odour of iodoform was no longer perceptible. The substance had the melting point 66-68°.

With bromoform exactly the same method was followed. But as it is highly volatile, the excess of it was easily removed from the yellow compound, which had the melting point 66-68°.

When the reaction mixture in alcoholic solution was heated to 210-220° in a sealed tube for several hours, a tarry resinous mass was obtained; it was filtered off the mother liquor and dissolved in benzene. Addition of alcohol to the benzene solution gave the same tarry precipitate but not the yellow powder. The alcoholic mother liquor, on concentration, gave needle-shaped shining crystals which had the sharp melting point 62°. Analysis proved this product to be the alcoholate of the compound described above, conforming to the formula $CH(Biaz)_3 + 2C_2H_5OH$. The yield of the crystalline product was very poor, evidently most part of the product had become resinified owing to the high temperature employed.

No reaction took place when chloroform was refluxed with the mercaptide in alcoholic solution. The mixture was therefore heated in a sealed tube as in the case of bromoform. The filtrate, separated from the resinified mass, gave, on evaporation, crystals of the alcoholate having the melting point 62°.

It is thus evident, that iodoform and bromoform are more reactive towards the mercaptide than chloroform.

Result of analysis :—

(1) The yellow powder from iodoform, m.p. 66-68°.

0·0953 gave 0·1502 CO_2 and 0·0202 H_2O.

0·0876 gave 0·2087 $BaSO_4$

0·075 gave 8.3 c. c. N_2 at 24° and 760 m. m. pressure.

Found	Calc. for $CH(Biaz)_3$
C 42·98	43·60
H 2·33	2·76
S 41·19	41·80
N 12·50	12·21

The yellow powder from bromoform gave $C=43\cdot1$, $H=2\cdot03$; $N=12\cdot59$.

The alcoholate :

Result of analysis :—

0·1520 gave 0·2532 CO_2
0·1101 gave 11·4 c.c. N_2 at 32° and 760 m.m. pressure
0·1260 gave 0·3532 $BaSO_4$.

Found	Calc. for $CH(Biaz)_3 + 2C_2H_5OH$
C 45·23	44·6
N 11·26	10·85
S 38·48*	37·50

K-Biaz and Nitro-chloroform.

The alcoholic solution of the parent substances was refluxed as before. A bulky yellow precipitate was obtained ; after decanting off the mother liquor it was washed with alcohol and triturated with water in a mortar to remove the potassium chloride. It was obtained pure by crystallising from hot benzene. Yellow shining crystals were obtained ; melting point 128-129°. The reaction was almost quantitative as was proved by weighing the potassium chloride formed.

Result of analysis :—

0·1140 gave 0·1727 CO_2 and 0·0307 H_2O.
0·0948 gave 11·4 c. c. N_2 at 32° and 760 m. m. pressure.

Found	Calc. for $NO_2C(Biaz)_3$
C 41·31	41·97
N 13·18	13·37

K-Biaz and tribromoresorcinol.

The alcoholic solution of the parent substances was refluxed as before. The amorphous powder, obtained after the reaction, was

* It may be necessary to point out that in these compounds, the sulphur atom is linked both to a fatty as also to an aromatic and cyclic residue. Carius' method gave the result unusually low owing to the formation of sulphonic acid. The excess of nitric acid was, therefore, neutralised with sodium carbonate and evaporated to dryness and fused in a silver dish. The "melt" was several times evaporated with hydrochloric acid before the addition of barium chloride. Owing to the presence of a large amount of sodium chloride and unacted upon sodium nitrate, the result generally comes out a little high.

freed from potassium bromide by water and was dried. It was dissolved in a mixture of alcohol and carbon bisulphide; on evaporation, shining crystals melting at 166° were obtained.

Result of analysis :—

0·0881 gave 0·1458 CO_2 and 0·0236 H_2O.

0·0724 gave 0·1897 $BaSO_4$.

	Found	Calc. for $(OH)_3C_6H(Biaz)_3$.
C	45·13	46·03
H	2·97	2·30
S	36·00	36·55

K-Biaz and Benzal-chloride.

The components in alcoholic solution were refluxed for about an hour. The crystalline product had the melting point 59-62°.

Result of analysis :—

0·0708 gave 0·1321 CO_2 and 0·0193 H_2O

0·0849 gave 8·6 c. c. N_2 at 22° and 760 m. m. pressure.

0·0974 gave 0·2550 $BaSO_4$.

	Found	Calc. for $C_6H_5CH(Biaz)_2$
C	50·87	51·11
H	3·03	2·96
N	11·02	10·40
S	35·83	35·55

K-Biaz and Ethylene dibromide.

After refluxing in alcoholic solution as usual the insoluble product was freed from potassium bromide by means of water and dried. This was then dissolved in ether. On evaporation, shining crystals melting at 94° began to appear.

Result of analysis :—

0·1644 gave 0·2183 CO_2

0·0812 gave 6·6 c.c. N_2 at 30° and 760 m.m. pressure.

0·1303 gave 0·0715 AgBr

	Found	Calc. for $BrCH_2CH_2(Biaz)$.
C	36·21	36·03
N	8·97	8·40
Br	23·35	24·02

K-Biaz and Monochloro acetic acid.

The reaction takes place even in the cold on mixing up the components in aqueous solution and is completed on heating. The product crystallises out from boiling water in silky needles of melting point 145°.

Result of analysis :—

0.1279 gave 0·1935 CO_2 and 0·0414 H_2O

0·1580 gave 0·3495 $BaSO_4$

0·1697 gave 15·00 c.c. N_2 at 24° and 760 m.m. pressure.

	Found	Calc. for CO_2HCH_2(Biaz).
C	41·26	42·06
H	3·60	2·81
N	9·99	9·86
S	30·38*	33·81

It is of interest to note that while monochloroacetic acid behaves like an ordinary chlorinated body, dichlor-, and trichlor-acetic-acids, on the other hand, behave like "strong acids" such as hydrochloric acid[*]; i.e. they simply reproduce the original mercaptan.

Potassium salt of thiobiazoledisulphydrate and nitro-chloroform.

The components were refluxed with alcohol for several hours. After some time, nitrous fumes were evolved and a yellow precipitate was obtained. On cooling, it was filtered off the mother liquor and washed with alcohol to remove the adhering nitrochloroform. The powder was triturated in a mortar with water to remove the potassium chloride formed during the reaction. The yellow powder was found to be insoluble in nearly a dozen of ordinary solvents and even in mixtures of some of these. Two different preparations had, however, the same melting point (166-168°) and the same percentage composition.

Result of analysis :—

0·1734 gave 0·1274 CO_2

0·0853 gave 14·00 c.c. N_2 at 31° and 760 m.m. pressure.

0·0899 gave 0·3907 $BaSO_4$.

	Found	Calc. for $(C_2N_2S_3)_3C_2O$.
C	20·04	19·83
N	18·03	17·40
S	59·68	59·50

* For explanation of the low result of sulphur, see foot note ante p. 52.

Mercuri-Sulphoxy Chloride.

Chain Compounds of Sulphur.

Part V.

BY

Prafulla Chandra Ráy.

The close analogy between mercuric chloride and nitrite has been found to hold good throughout the investigations carried on from 1918 onwards. Thus, whilst mercuric *nitrate* with sodium sulphate at once gives an abundant yellow precipitate of the oxy-sulphate, known as the *turpeth mineral*—the chloride and the nitrite fail to give it. (Trans. Chem. Soc. 1897, 1103). The explanation lies in the fact that these latter salts are very feebly ionised in solution and thus have no tendency to yield basic compounds. Ammonia, amines and even a class of alkaloids have been found to behave towards mercuric nitrite similar to the chlorides (Trans. 1913, *103*, 3; Trans. 1917, *111*, 507).

The substituted thiocarbamides, thiocarbimides, thiobenzamide etc., have been shown to give rise to a purely inorganic sulphoxy-nitrite of the empirical formula $[3(SHgNO_2)HgO]_2$, which is in reality a chain compound containing six atoms of sulphur in the link (Trans. 1917, *111*, 104).

Recently, an attempt was made to isolate the radicle $(SHgCl)_2$, which would be the analogue of mercuri-iodo-sulphide $(SHgI)_2$ l.c. 109, by treating mercuric chloride with some typical thio-bodies named above, as also thioacetic acid and ammonium dithio-carbamate. It was expected that in each case the radical $(SHgCl)$ would get detached from the parent body and lead an independent existence. This expectation has been realised, but in a qualified sense. The radical $SHgCl$, no sooner it separates out, assumes the from $[3(SHgCl)HgO]_2$, which is the exact analogue of the oxy-nitrite. It has been shown already that the complex nitrite containing several NO_2 groups by the elimination of N_2O_3 (i.e. $NO + NO_2$) readily yields the oxy-nitrite.

At the first sight, it is not easy to understand how the chloride also would give birth to an oxy-salt. The explanation lies ready at hand when it is borne in mind that here water takes part in the reaction; $[3(SHgCl) HgO]_2$ is formed thus:

$$
\begin{array}{cccc}
HgCl & HgCl & HgCl & HgCl \\
| & | & | & | \\
ClHg.S.S &\!-\!-\!-\!-\!S\!-\!\!-\!\!-S\!-\!\!-\!\!-S.S.HgCl \\
& & & | \\
\overline{Hg}\,\overline{ClH}\,\overline{O}\,\overline{HCl} & \overline{ClH}\,\overline{O}\,\overline{HCl}\,\overline{Hg}
\end{array}
$$

i.e.,

$$
\begin{array}{cccc}
HgCl & HgCl & HgCl & HgCl \\
| & | & | & | \\
ClHg.S.S\!-\!\!-\!S\!-\!\!-\!\!-S\!-\!\!-\!\!-S.S.HgCl + 4HCl. \\
| & | & | & | \\
Hg\!-\!\!-\!O & O\!-\!\!-\!Hg
\end{array}
$$
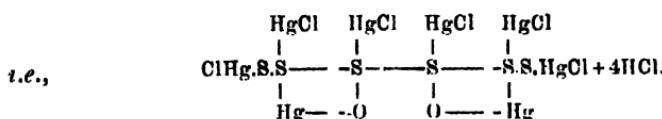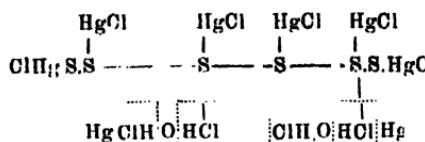
In other words, as soon as the radicle SHgCl separates out every three of this take up an additional molecule of mercuric chloride, *i.e.*, the radicles HgCl and Cl and a molecule of water simultaneously takes part in the reaction, four molecules of hydrochloric acid are generated and the oxygen atom forms the connecting link between the mercury atom and the neighbouring sulphur atom and the two symmetrical complexes coalesce into a single molecule.

It is necessary to point out here that whilst the radicle $SHgNO_2$ has often a tendency to part company with the parent body, the radicle SHgCl often prefers, on the other hand, to remain attached to it. Thus, thiocarbamide when acted upon by mercuric chloride yields the compound $(NH_2–C:NH–SHgCl)$ HCl, which is actually a *hydrochloride* as has been shown in a previous communication. If, however, thiocarbamide be converted into its diacetyl derivative and then treated with mercuric chloride, the molecule undergoes a break up with the detachment of the radicle SHgCl and formation of the oxychloride; S–diphenythiocarbamide also behaves similary.* Evidently, the introduction of the negative radicles acetyl and phenyl neutralises the basic character of the compound due to the presence of an amino- and imino group and deprives it of the power of the formation of a hydrochloride in which character alone it is stable.

---

* P. K. Sen working in this laboratory has prepared diacetyl thiocarbamide and has already shown that both this compound and s-diphenyl thiocarbamide yield with mercuric chloride the oxy-chloride $[3(SHgCl)HgO]_2$.

It is a characterestic property of mercaptans, real and potential, that with mercuric nitrite and chloride they yield the nitrite mercaptide and chloro-mercaptide respectively. Thioacetic acid, though an acid, contains the group SH and as such behaves like a typical mercaptan.

Thus :

$$CH_3-\overset{\overset{O}{\|}}{C}-SH \xrightarrow{HgCl_2} CH_3\overset{\overset{O}{\|}}{C}-SHgCl$$
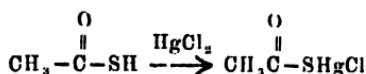
As soon as this compound is formed it acts upon a molecule of water.

$$CH_3-\overset{\overset{O}{\|}}{C}-\overset{|}{\underset{H}{}}-SHgCl$$
$$OH$$
$$CH_3-C-SHgCl$$
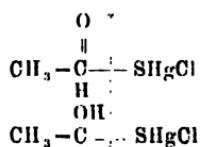
and the scission takes place as shown by the dotted line, a molecule of acetaldehyde and acetic acid being formed.

Ammonium dithiocarbamate assumes the tautomeric form $NH=C<\overset{SNH_4}{\underset{SH}{}}$, both the radicles -SH and -SNH$_4$ with mercuric chloride yield -SHgCl, which separates out and assumes the oxychloride form.

Allyl thiocarbimide $C_3H_5N:C:S$ simply takes on the components of mercuric chloride and $C_3H_5N:C:SCl-HgCl$ is temporarily formed, the sulphur atom becoming tetravalent. As this confignration is unstable, a rupture takes place along the line of least resistance, the radicle $=S<\overset{Cl}{\underset{HgCl}{}}$ decomposes into the stable radicle -SHgCl and chlorine ; whilst the organic portion of the complex $RN:C<$ with a molecule of water yields a primary amine and carbon monoxide.

Experimental.

Method of preparation :—The thio-body in aqueous or alcoholic solution as the case may be was added by means of a pipette in a thin stream to an aqueous solution of mercuric chloride under vigorous stirring, care being taken that the latter was always in large excess.

A granular white preocipitate was obtained which was washed first with water and then with alcohol and finally dried in a vacuum over sulphuric acid. A special precaution is necessary in the case of mustard oil. If an alcoholic solution of it is added to an aqueous solution of mercuric chloride the white precipitate is obtained, but, at the same time heavy oily globules begin to settle down at the bottom and it is not always easy to separate them from the sulphoxy-chloride. It is best to add the alcoholic solution of the mustard oil to an alcoholic solution of mercuric chloride. The mixture remains clear but on copious dilution with water and stirring, the white precipitate begins to put in an appearance. The reaction mixture is allowed to stand over night and the produce collected and treated as before.

The interaction of each of the above mentioned thio-bodies with mercuric chloride was repeated several times and the composition throughout was found to be identical. It is therefore not necessary to give the analysis of each preparation ; that of one or two typical ones are given below.   •   •

Mercuric chloride and thioacetic acid.

Result of analysis :—

0·2587 gave 0·2370 HgS ;

0·1465 gave 0·0625 AgCl and 0·1070 BaSO$_4$ by fusion with NaNO$_3$ and Na$_2$CO$_3$.

	Found	Calc for Hg$_4$S$_3$Cl$_3$O
Hg	78·98	78·58
Cl	10·55	10·46
S	10·03	9·43

Mercuric Chloride and mustard oil.

Result of analysis :—

0·3402 gave by distillation with copper powder 0·2631 Hg.

0·2795 gave 0·1285 AgCl and 0·1818 BaSO$_4$.

Found :—Hg = 77·34 ; Cl = 11·37 ; S = 8·93.

The absence of carbon was shown by combustion analysis.

Mercury Mercaptide Nitrites and their Reaction with the Alykyl Iodides. Part VI. Chain Compounds of Sulphur—*contd.*

BY

Prafulla Chandra Ray

AND

Prafulla Chandra Guha.

In part IV of the present series of investigations, it has been shown that thiobiazole disulphydrate gives by interacting with mercuric nitrite a series of sulphoxy compounds in which two, three and four molecules respectively, of the dimercaptan are condensed. In the present investigation, the polysulphonium (viz. tetra-, hexa-, octa-, and dodeca) compounds will mainly be dealt with, which are formed by the action of alkyl iodides upon these sulphoxy compounds as also upon a new hexanuclear sulphoxy one, formed by the condensation of six molecules of the dimercaptan.

The sulphoxy bodies may be included under the general formula $(C_2N_2S_3)$ $x Hg_2O$; where $x = 2, 3, 4$ or 6. A condensation product of five molecules has not as yet been obtained. It is not easy to explain why in one operation the value of x should be two and in others it should run up to six; possibly the concentration of the parent substances is the main determining factor. It has often been found that two preparations under similar conditions had identical composition. In the majority of cases the value of x has been found to be three occasionally two and four, and only rarely six.

The preparations could not have been admixtures because, each of them strictly conformed to a definite formula. But the must convincing proof of these compounds being of definite composion is afforded by their deportment towards the alkyl iodides. These sulphoxy bodies behave exactly like mercaptide nitrites and yield, as a rule, the corresponding sulphonium derivatives and in a few cases those with a fewer number of nuclei. The reducing action of

the alkyl iodide removes the oxyen atom of the sulphoxy ring and the bonds being thus snapped, an open chain compound is formed ; thus :

$$S-R''-S-S-R''-S-S-R''-S \quad Rl$$
$$Hg \text{———}O\text{———}Hg \longrightarrow IHg-S-R''-S-S-R''-S-S-R''-S-HgI$$
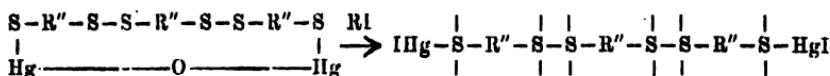
The six sulphur atoms of the chain now become quadrivalent by taking up the components of alkyl iodide.

In this manner, a series of tetra—, hexa—, octa— and dodeca-sulphonium compounds have been prepared. Each of these with the exception of propyl and butyl derivatives is characterised by its crystalline character and moreover its successive crops have been found to have the same melting point; the possibility of their being mixtures is thus precluded.

Another interesting point is the shifting of the double bonds; thus :

$$\begin{array}{c} N-N \\ \| \quad \| \\ -C \quad C- \\ V \\ S \end{array}$$
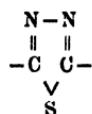
takes up two alkyl radicles and is changed into

$$\begin{array}{c} N=N \\ | \quad | \\ -C \quad C- \\ |V| \\ R\,S\,R \end{array}$$
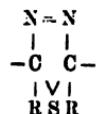
where R=Me, Et, Pr, Bu. As a rule, this is confined only to one nucleus. Evidently we have here an extension of Thiele's theory to nitrogen compounds.

In one isolated instance and that in the case of interaction with methyl iodide, instead of there being a shifting of the double bonds, both the pairs of nitrogen and carbon atoms throughout the molecule were saturated by taking up additional methyl radicles ; thus :

$$\begin{array}{ccc} N-N & Me & Me-N-N-Me \\ \| \quad \| & \longrightarrow & | \quad | \\ -C \quad C- & & -C \quad C- \\ V & & |V| \\ S & & Me\,S\,Me \end{array}$$

EXPERIMENTAL.

$$\textit{Hexanuclear sulphoxy compound} \quad \begin{array}{c} S-(C_2N_2S)_6S_{16}-S \\ | \qquad\qquad\qquad | \\ Hg\text{———}O\text{———}Hg \end{array}$$
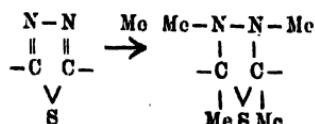
The compound was associated with 7 H_2O.

0·1236 gave 0·049 HgS. Hg=28·52

0·1109 „ 0·3215 $BaSO_4$. S=39·82

0·1715 „ 18 c.c. N_2 at 30° and 780 mm. N=11·59

Theory requires, Hg=27·97; S=40·28; N=11·75

II

INTERACTION WITH THE ALKYL IODIDES.

General method of preparation :

The above sulphoxy derivatives were refluxed on a water bath with alkyl iodides for several hours and allowed to stand overnight, sometimes a crystalline mass and sometimes a heavy dark brown oil settled at the bottom; the excess of alkyl iodide was decanted or boiled off and the former dissolved in the minimum quantity of acetone and filtered off from the insoluble matter whenever necessary. On adding ether to the filtrate, a pale yellow, mealy crystalline precipitate was obtained. This process was repeated to further purify the product.

BINUCLEAR CONDENSATION.

(1) $\begin{array}{c} S-(C_2N_2S)_2S_2-S \\ | \qquad\qquad | \\ Hg\text{———}O\text{———}Hg \end{array}$ *and methyl iodide: formation of Dithiobiazole tetrasulphonium methyl mercuri-iodide.*

$$\begin{array}{c} \text{Me N—N Me Me N=N Me} \\ | \quad \| \quad \| \quad | \quad | \quad | \quad | \quad | \\ IHg\text{—}S\text{—}C \quad C\text{—}S\text{—}S\text{—}C \quad C\text{—}S\text{—}HgI \\ | \quad V \quad | \quad | \quad |V| \quad | \\ I \quad S \quad I \quad I \text{ MeS Me } I \end{array}$$
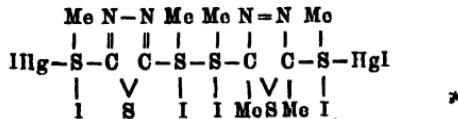
The product had the m.p. 101-102°

0·2186 gave 0·0567 Hg and 0·1971 AgI. Hg=25·94; I=49·23

0·1610 „ 0·0480 CO_2. C=7·75

$C_{10}H_{18}N_4S_6Hg_2I_6$ requires Hg=25·84; I=49·33; C=8·18

(2) The corresponding ethyl iodide compound was sparingly soluble in acetone and was therefore purified by crystallisation from the boiling solvent, m.p. 107°

Prep. I

0·2674 gave 0·0647 Hg and 0·2138 AgI.   Hg = 24·19 ; I = 43·2*

0·2343   „   0·0967 CO_2.   C = 11·25

0·2049   „   6·4 c.c. N_2 at 24° and 760 mm.   N = 3·53

Prep. II

0·3600 gave 0·0885 Hg and 0·1232 gave 0·0516 CO_2.
    Hg = 24·58 ; C = 11·42

$C_{16}H_{30}N_4 S_6 Hg_2 I_6$ requires, Hg = 24·51 ; I = 46·69 ; C = 11·76 ;
    N = 3·43 †

The corresponding binuclear tetrasulphonium compound with n. propyl and n. butyl iodides did not yield crystalline products, but only a dark brown treacle-like pasty mass, which was purified by repeated precipitation by ether from acetone solution. It has already been shown that with the simpler mercaptides, these higher n. alkyl iodides gave crystalline products with sharp m.p. (Cf. T. 1916, *109*, 606). It is very likely that in these particular cases the alkyl radicles are converted into the iso-varieties, from the consideration that isoalkyl iodides always gave such pasty masses (l.c. 607).

(3) *The corresponding tetrasulphonium propyl compound :*

0·3407 gave 0·0814 Hg and 0·2821 AgI.   Hg = 23·89 ; I = 44.74

0·2271   „   0·1170 CO_2.   C = 14·05

$C_{22}H_{42}N_4S_6Hg_2I_6$ requires Hg = 23·31 ; I = 44·40 ; C = 15·38

(4) The butyl compound :

0·3675 gave 0·0800 Hg and 0·2839 AgI.   Hg = 21·77 ; I = 4175

0·1973   „   0·1278 CO_2.   C = 17·66

0·1280   „   3·6 c.c. N_2 at 30° and 760 mm.   N = 3·11

$C_{26}H_{54}N_4S_6Hg_2I_6$ requires Hg = 22·22 ; I = 42·33 ; C = 18·60 ;
    N = 3·11

---

*As a large quantity of copper powder has to be used and as the process is a tedious one, the iodine and the sulphur sometimes come out low (see T.1916, *109*, 611)

† As these polysulphonium compounds are non-oxylic every element in them admits of its percentage being determined by direct analysis; it has not been however deemed essential to do so in every individual case.

TRINUCLEAR CONDENSATION.

$$\begin{array}{c} S-(C_2N_2S)_2S_4-S \\ | \qquad\qquad | \\ Hg\text{——}O\text{——}Hg \end{array}$$

(5) and methyl iodide : formation of Tri-thiobiozole hexasulphonium methyl mercuri-iodide.

$$\begin{array}{ccccccccccccc} & Me & N-N & Me & Me & N-N & Me & Me & N-N & Me & \\ & | & \| \quad \| & | & | & \| \quad \| & | & | & | \quad | & | & \\ IHg-S-C & & C-S-S-C & & C-S-S-C & & C-S-HgI \\ & | & V \quad | & | & | & V \quad | & | & | & V | & | & \\ & I & S \quad I & I & S & | & I & I & MeS & Me & I \end{array}$$

The product on refluxing the parent sulphoxy compound with methyl iodide was an " oil;" it was dissolved in hot acetone, which on cooling deposited a crystalline mass. This on recrystallisation from hot acetone yielded needle shaped crystals m.p. 101-102°, it was in fact the same compound as described on p. 61 (compound No. 1) The original mother liquor on concentration gave two successive crops m.p. 85-86°., the analysis of which is given below.

0·2947 gave 0·0635. Hg = 21·55

0·1480 „ 0·0470 CO_2. C = 8·66

$C_{14}H_{24}N_6S_9Hg_2I_6$ requires Hg = 20·20 ; C = 8·48

(6) The compound with methyl iodide in which all the double bonds are saturated, viz ;

$$\left[\begin{array}{c} Me-N-N-Me \\ Me \quad | \quad | \quad Me \\ | \qquad\qquad | \\ S-C \quad C-S- \\ | \quad | \quad V | \quad | \\ I \quad MeS \quad Me \quad S \end{array}\right]_2 Hg_2I_2$$
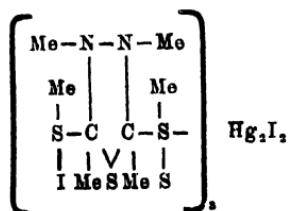
The method of preparation and purification was the same as in the case of the preceding compound. It was a white crystalline substance, m. p. 94°

0·3717 gave 0·3678 Hg and 0·3075 $BaSO_4$. Hg = 18·24 ; S = 11·36

0·2213 „ 0·0882 CO_2 and 0·0506 H_2O. C = 13·15 ; H = 2·54

$C_{24}H_{54}N_6S_9Hg_2I_6$ requires Hg = 18·78 ; S = 13·52 ; C = 13·52 ; H = 2·53

Formation of this type of compound has been noticed only in this one instance.

(7)   *The hexasulphonium propyl compound :*

It conforms to the ordinary type (cf. No. 5).

0·2912 gave 0·0540 Hg and 0·2404 AgI.   Hg=18·54; I=44·61

0·1467 „   0·0886 CO_2.   C=16·47

$C_{30}H_{56}N_6S_9Hg_2I_8$ requires Hg=18·14; I=46.1 ; C=16·3

As will be noticed, the trinuclear sulphoxy compound gives compounds No. 1 and 5 with methyl iodide ; of course, the latter in major proportion.   This tendency towards the formation of the di-nuclear tetrasulphonium compounds from the higher nuclear sulphoxy bodies is particularly noticeable in the case of interaction with ethyl iodide. With this alkyl iodide only the di-nuclear sulphonium compound (No. 2) is always obtained, even from tri-and tetra-nuclear sulphoxy derivatives.   In all these cases of formation of a lower member from the higher sulphoxy compounds, a dark brown pasty substance with a penetrating odour and lachrymal properties was always formed, which resisted all attempts at purification.

Hexanuclear Condensation.

(8)
$$\begin{array}{c} S-(C_2N_2S)_6S_{10}-S \\ | \qquad\qquad\qquad | \\ Hg \quad --O------ Hg \end{array}$$
and ethyl iodide : *formation of Hexa-thiobiazole dodecasulphonium ethyl mercuri-iodide.*

$$\left[\begin{array}{cccc} Et & N-N & Et \\ | & \| \ \| & | \\ -S-C & & C-S- \\ | & V & | \\ I & S & I \end{array}\right]_6 \qquad \begin{array}{cccc} Et & N=N & Et \\ | & | \ \ | & | \\ -S-C & & C-S-HgI \\ | & |V| & | \\ I & EtSEt & I \end{array}$$
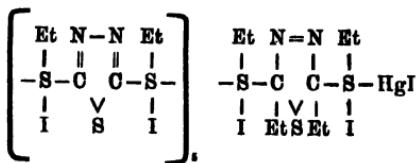
It had the m. p. 90-91°

0·3531 gave 0·3269 AgI. I=50·03

0·2008 „   7 c. c. N_2 at 27° and 760 mm.   N=3·90

0·1548 „   0·0746 CO_2.   C=13·14

$C_{40}H_{70}N_{12}S_{18}Hg_2I_{14}$ requires I=51·20; N=4·84 C=13·82.

It will thus be seen that the type persists throughout, in that the alteration in the position of the double bond is limited to only one nucleus of the chain,

. In the previous communications the compounds hitherto described were tentatively classed under the sulphonium group, though no direct proof could be adduced for adopting this view. One of the purest compounds of these series, *viz.* Me. $EtS_2.HgI_2$. EtI ; (T. 1916-*109*, 606) was selected for molecular weight determination in acetone solution by the ebullioscopic method ; the value obtained was 712, that required by theory being 718. It is thus evident that the constitution is atomic ; in fact, Hilditch and Smiles threw out a suggestion in the case of monosulphonium compounds that these are not molecular. (T. 1907, *91*, 1396). A study of the physical properties of the interesting polysulphonium compounds treated of in this paper is being undertaken, which, it is hoped, will throw additional light upon their constitution.

Mercury Mercaptide Nitrites and their Reaction with the Alkyl Iodides Part VII. Chain Compounds of Sulphur.

By

Prafulla Chandra Ráy.

The present investigation deals with the chain compounds of hexavalent sulphur. On treating the product of interaction of thiocarbamide with mercuric nitrite, *viz.*, the sulphoxynitrite,

$$NH_2 \cdot C(:NH) - \underset{\underset{O}{\overset{\parallel}{V}}}{\overset{\overset{HgNO_2}{|}}{S}} - Hg$$
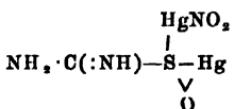
(Trans. 1917, *111*, 102) with ethyl iodide a yellow crystalline substance was obtained. It was found to be completely soluble in acetone. On purification by precipitation with ether it had the m.p. 98—100°. Analysis proved it to conform to the formula $Et_2S_2, EtI, 2HgI_2$; in other words, it is a member of the disulphonium series already described (Trans. 1916, *109*, 134) with an additional molecule of mercuric iodide in combination, to which should be assigned the constitutional formula

$$\underset{\underset{I}{\overset{\diagup \diagdown}{I}} \quad I}{\overset{\overset{Et \quad HgI \quad HgI}{\diagdown \diagup \quad |}}{Et - S \underline{\quad\quad} S - Et}}$$
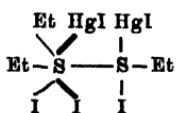
one of the sulphur atoms in the chain becoming hexavalent. The formation of this compound suggested the possibility of the direct conversion of. *all* the members of the series, $RR'S_2$, $R'I$, HgI_2 into $RR'S_2$, $R'I$, $2HgI_2$. This anticipation has been realized with one notable exception. When the alkyl radical happens to be methyl, combination with an extra molecule of mercuric iodide does not take place. The presence of the radical ethyl, on the other hand, favours the combination. Thus in the above series where R = Me and R' = Pr and Bu the extravalencies of the sulphur atom are not revived; but if R' happens to be ethyl, this anomaly disappears. To what extent ethyl favours the increase in valency will be evident from a typical case.

When the ethyl mercaptide nitrite (Et-SHgNO$_2$) is treated with methyl iodide, by an interchange of the radical we get EtMeS$_2$, HgI$_2$, MeI (l.c. 603); but in this case although there are two methyl groups the presence of a single ethyl radical is sufficient to counteract the prejudicial influence of the former and the compound EtMeS$_2$, MeI, 2HgI$_2$ is readily formed.

The marked genetic affinity of the radical ethyl for sulphur and its influence in the increment of its valency is further evidenced by the fact that a compound of the empirical formula Et$_4$S$_2$, 2EtI, HgI$_2$ has also been obtained from ethyl sulphide by its interaction with ethyl iodide and mercuric iodide. On repeating Smiles' experiment (Trans. 1900, 77, 161) under slightly altered conditions, with a view to ascertain the maximum valency of sulphur, it was noticed that while the main product was (C$_2$H$_5$)$_3$SI, HgI$_2$, as found by the above chemist, there was always a considerable amount of a shining crystalline substance, practically insoluble in cold acetone. As it had a sharp m.p. when crystallized out from hot acetone, it was subjected to analysis, which established the formula given above. What evidently happens is that under the joint action of mercuric iodide and ethyl iodide or rather their ions, the divalent sulphur atoms of two adjacent molecules of ethyl sulphide become hexavalent, by their dormant valencies becoming active and they coalesce into a single molecule thus :

$$\begin{array}{ccc} \text{Et} & \text{Et} & \text{HgI} \quad \text{Et} \\ \diagdown\diagup & & \diagdown\diagup \\ \text{Et}-\text{S}\!\!\!-\!\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\text{S}-\text{Et} \\ \diagup\diagdown & & \diagup\diagdown \\ \text{I} \quad \text{I} & & \text{Et} \quad \text{I} \end{array}$$
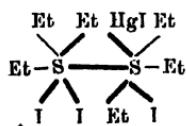
It is remarkable that in the above reaction if ethyl iodide be substituted for methyl, propyl, or butyl iodide the products formed in each individual case is found to be completely soluble in acetone and conforms to the general formula Et$_2$RISHgI$_2$, but no product of the fusion of two ethyl sulphide molecules is formed. The differential property of ethyl as contradistinguished from other alkyl radicles is thus brought into relief.

It was expected that the general method of the preparation of the disulphonium compounds already described, viz. the treatment of ethyl mercurimercaptite nitrite EtSHgNO$_2$ with ethyl iodide should also yield the chain compound with both the sulphur atoms as hexavalent. This expectation has also been fulfilled. The latter compound is formed in such a small quantity that on previous

occasions its formation was overlooked. It has already been shown that ethyl disulphide, ethyl iodide and mercuric iodide also combine directly to yield the disulphonium compound, Et_2S_2, HgI_2, EtI. (l.c.). Recently this preparation has been repeated and it has been found that the hexavalent *di*sulphur compound is also formed in considerable quantity along with the former. It is thus evident that both the tetra-as also the hexavalent chain compound of sulphur is formed simultaneously.

It is of interest to note that Smiles and Hilditch who treated an acetone solution of equimolecular proportions of ethyl disulphide and mercuric iodide with ethyl iodide obtained *diethyl thioethyl sulphonium dimercuric sulphide.* $(C_2H_5)_3S_2I$, $2HgI_2$ (Trans. 1907, *91*, 1396). It is evidently the same compound as has been described above.

An explanation may be offered as to why it is that in the first series of compounds only one of the two atoms of sulphur can behave as hexavalent ; here the tetravalent sulphur being already weighted with the heavy load of the ions of HgI′ and I′ has lost the capacity to take up additional charge ; in other words, to acquire the maximum valency. In the solitary instance, however, in which both the sulphur atoms happen to be hexavalent it will be noticed that there is only one set of HgI′ and I′ions ; the sulphur atom which have these latter has got attached to it three additional com- paratively light Et radicals, whereas, the other sulphur atom not having to bear the load of the heavy HgI′ is in a position to take up three Et's and two I's. Facts are already known which go to support the view that the maximum valency of an element is often conditional upon the load of the radicles. The author has in a previous communication shown that platinum when attached to the radical of phenyl dithiobiazolinesulphydrate behaves as trivalent. It is none the less inexplicable why the light radical methyl should stand in the way of one of the atoms of sulphur attaining its maximum valency. The anomalous behaviour of the first member of the alkyl series is, however, well known.

EXPERIMENTAL.

General method of Preparation of the series R_2S_2, RI, $2HgI_2$ has already been incidentally described. These members are readily obtained by dissolving the corresponding disulphonium compound in acetone and adding mercuric iodide to the solution till no more is

taken up. The thick golden yellow liquid is decanted off the un-dissolved iodide. On adding ether to its acetone solution a copious deposit of yellow mealy crystals is obtained. Solution in acetone and precipitation by ether is repeated till the product gives a fairly sharp melting point. It has been found that in some instances as specially in the case of methyl sulphonium compound, Me_2S_2, MeI, HgI_2, the acetone solution at first takes up a considerable quantity of mercuric iodide but purification by the above process gradually removes all the mechanically held mercuric iodide, which evidently remains obstinately adhering to it.

Result of Analysis.

(1) (*a*) From the sulphoxy nitrite derivative of thiocarbamide : m. p. 98°

0·3656 gave 0·1250 Hg, 0·3540 AgI and 0·1029 $BaSO_4$.
$Hg = 34·19$; $I = 52·32$; $S = 5·50$

0·2442 gave 0·0570 CO_2 and 0·0356 H_2O. $C = 6·37$; $H = 1·62$
Calc. for Et_2S_2, EtI, $2HgI_2$, $Hg = 33·73$; $I = 53·54$; $S = 5·4$;
$C = 6·07$; $H = 1·26$.

(*b*) Direct synthesis *i.e.*, by the direct union of HgI_2 with Et_2S_2 EtI, HgI_2 : m. p. 100—101°

0·2118 gave 0·0711 Hg. $Hg = 33·57$
0·2118 gave 0·0532 CO_2 and 0·0352 H_2O. $C = 6·85$; $H = 1·71$.

(2) $MeEtS_2$, EtI, $2HgI_2$: m. p. 38—40°

0·2094 gave 0·0452 CO_2 and 0·0466 H_2O. $C = 5·87$; $H = 2·47$
0·3840 gave 0·1236 Hg and 0·3768 AgI. $Hg = 32·19$; $I = 53·03$
Calc. $C = 5·12$; $H = 1·11$; $Hg = 34·14$; $I = 54·19$.

(3) $EtMeS_2$, MeI, $2HgI_2$ m. p. 50—55° (not sharp)

0·2467 gave 0·2483 AgI and 0·0840 Hg. $Hg = 34·05$; $I = 54·39$
0·1428 gave 0·0286 CO_2 and 0·0211 H_2O. $C = 5·46$; $H = 1·64$
Calc. $Hg = 34·55$; $I = 54·84$; $C = 4·14$; $H = 0·95$.

(4) $EtPrS_2$, PrI, $2HgI_2 + CH_3COCH_3$. This compound contains one molecular proportion of acetone : m. p. 30—31°

0·4410 gave 0·1400 Hg. $Hg = 31·75$
0·1629 gave 0·0586 CO_2 and 0·0352 H_2O. $C = 9·81$; $H = 1·97$.

(5) $EtBuS_2$, BuI, $2HgI_2 + 1\frac{1}{2} CH_3COCH_3$. The substance had the consistency of treacle and had $1\frac{1}{2}$ mols. of acetone combined with it.

0·1340 gave 0·0649 CO_2 and 0·0363 H_2O. C=13·21; H=3·01
0·3879 gave 0·1153 Hg and 0·3272 AgI. Hg=29·72; I=45·58
Calc. C=13·10 ; H=2·41 ; Hg=30·10 ; I=47·79.

(6) The compound containing both the sulphur atoms as hexavalent:
it was very sparingly soluble in cold acetone but fairly soluble in
the boiling solvent, m. p. 146—147°

Prepared from Et_2S :

0·1930 gave 0·1068 CO_2 and 0·0538 H_2O. C=15·09 ; H=3·09
0·2991 gave 0·063 Hg and 0·2916 AgI. Hg=21·06; I=52·69
0·4032 gave 0·8400 Hg and 0·2042 $BaSO_4$. Hg=20·86; S=6·96
Calc. for $Et_6S_2HgI_4$ C=15·22; H=3·17; Hg=21·14; I=53·70;
S=6·77.

MATHEMATICS

On the motion of two spheroids in an infinite liquid along the common axis of revolution.

By

Bibhutibhushan Datta.

It is well-known that the problem of the motion of two spheres in an infinite liquid along the line joining their centres has been completely solved by various investigators. But the corresponding problem for two spheroids or ellipsoids has remained unsolved up to now; the only previous writer to attempt it with some measure of success being Prof. Karl Pearson.[*]

The object of this paper is to show how the problem can be solved in the case of two spheroids of revolution of *small* ellipticities, the motion of the solids being along their common axis of revolution. The method used is one of continued approximation.

In Art. 1, I investigate the approximate formal expressions for the velocity potential and the current function. In Art. 2, the first approximation to the velocity potential is obtained. Art. 3 gives two sets of algebraic equations from which, by continued approximations, we can determine the necessary constants and thus proceed to the complete solution. In Art. 4, I give the general values of the constants.

All the results in this paper are believed to be new.

I wish to express my indebtedness to Dr. Ganesh Prasad at whose suggestion I took up, and under whom I carried on, the investigation the results of which are embodied in this paper.

§ 1.
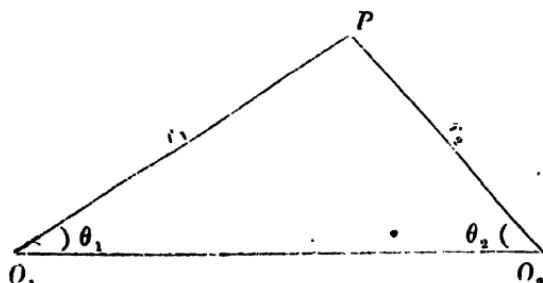
[*] "On the motion of spherical and ellipsoidal bodies in fluid media," Part II, (*Quarterly Journal of Pure and Applied Mathematics*, Vol. XX).

Let (r_1, θ_1) and (r_2, θ_2) be the polar co-ordinates of a point P referred to two points O_1 and O_2; let s be the distance between them. Then we shall take $\dfrac{d}{ds_{1\,2}}$ to denote differentiation in the direction of s measured from O_1 to O_2 and $\dfrac{d}{ds_{2\,1}}$ to denote differentiation in the opposite direction, so that $ds_{1\,2} = -ds_{2\,1}$.

The problem before us is to find a function ϕ satisfying the conditions :—

(i) $\nabla^2 \phi = 0$,

(ii) $\phi = 0$, at infinity,

(iii) $\dfrac{d\phi}{dn_1} = -u_1 \cos \left(\overset{\wedge}{n_1\ s_{1\,2}} \right)$ over the first solid.

(iv) $\dfrac{d\phi}{dn_2} = -u_2 \cos \left(\overset{\wedge}{n_2\ s_{2\,1}} \right)$ over the second solid,

where dn denotes an element of the normal to the surface drawn outward into the liquid, $\left(\overset{\wedge}{ns} \right)$ denotes the angle between the axis and the normal and u_1, u_2 are the velocities of the solids along the common axis of revolution.

Since the motion is symmetrical about the axis. we have another function ψ, called Stokes's current function,[*] and the boundary conditions (iii) and (iv) now take the forms.[†]

(iii^1) $\psi = -\tfrac{1}{2} u_1\, r_1^2 (1-\mu_1^2) + $ constant.

(iv^1) $\psi = -\tfrac{1}{2} u_2\, r_2^2 (1-\mu_2^2) + $ constant,
      where $\mu = \cos \theta$.

Let us assume[‡] that

(v) $\phi = \sum\limits_{n=1}^{n=m} \left[A_s\, \dfrac{d^s}{ds_{1\,2}{}^s} \left(\dfrac{1}{r_1} \right) + B_s\, \dfrac{d^s}{ds_{2\,1}{}^s} \left(\dfrac{1}{r_2} \right) \right],$

where m may be finite or infinite, A_s, B_s being arbitrary constants to be determined.

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[*] See Lamb's "Hydrodynamics," p. 117.   [†] Ibid, p. 122.

[‡] That this assumption is justifiable, will be clear from the actual values of the A'' and the B'' determined in the succeeding articles. The assumption was suggested by Prof. Pearson's paper.

The conditions (i) and (ii) are satisfied; it remains to determine the constants A_n and B_n so as to satisfy the conditions (iii) and (iv).

Now

$$r = s^2 + r_1^2 - 2sr_1\,\mu_1;$$

$$\therefore \quad \frac{1}{r_2} = \frac{1}{s} \cdot \sum_0^\infty \left(\frac{r_1}{s}\right)^p P_p(\mu_1),$$

s being greater than r_1.

Again

$$\frac{d}{ds_{21}} = \frac{dr_1}{ds_{21}} \cdot \frac{d}{dr_1} + \frac{d\mu_1}{ds_{21}} \cdot \frac{d}{d\mu_1}$$

the subject to be operated upon being a function of r_1 and μ_1.

Now

$$ds_{21} \cdot \mu_1 = -\,dr_1.$$

$$ds_{21} \cdot \sin\theta_1 = r_1\,d\theta_1;$$

therefore

$$\frac{d}{ds_{21}} = -\left(\frac{1-\mu_1^2}{r_1} \cdot \frac{d}{d\mu_1} + \mu_1 \frac{d}{dr_1}\right).$$

Hence

$$\frac{d}{ds_{21}}\left\{r_1^p P_p(\mu_1)\right\} = -r_1^{p-1}\left\{(1-\mu_1^2)\frac{dP_p(\mu_1)}{d\mu_1}\right.$$

$$\left. + p\mu_1 P_p(\mu_1)\right\} = -pr_1^{p-1} P_{p-1}(\mu_1),$$

therefore

$$\frac{d^n}{ds_{21}^n}\left\{r_1^p P_p(\mu_1)\right\} = (-1)^n \frac{p!}{(p-n)!}\, r_1^{p-n} P_{p-n}(\mu_1).$$

Hence

$$\frac{d^n}{ds_{21}^n}\left(\frac{1}{r_2}\right) = \frac{(-1)^n}{s^{n+1}} \sum_{p=n}^\infty \left(\frac{r_1}{s}\right)^{p-n} \frac{p!}{(p-n)!}\, P_{p-n}(\mu_1).$$

Further

$$\frac{d^n}{ds_{12}^n}\left(\frac{1}{r_1}\right) = \frac{(-1)^n n!}{r_1^{n+1}}\, P_n(\mu_1).$$

Hence

$$\phi = \sum_{n=1}^{n=m} \left[A_n \frac{(-1)^n n!}{r_1^{n+1}}\, P_n(\mu_1) + B_n \frac{(-1)^n}{s^{n+1}} \sum_{p=n}^\infty \frac{p!}{(p-n)!}\left(\frac{r_1}{s}\right)^{p-n}\right.$$

$$\left. \times\, P_{p-n}(\mu_1)\right].$$

Re-arranging, we have

$$(vi) \quad \phi = \sum_{n=1}^{n=m} \left[\frac{(-1)^n n!}{r_1^{n+1}} \ P_n(\mu_1) \right]$$

$$- \sum_{k=0}^{\infty} \left[\frac{r_1^k}{s^{k+1}} \ R_{k+1} \ P_k(\mu_1) \right],$$

where

(vii)

$$R_{k+1} = (k+1)\frac{B_1}{s} - (k+2)(k+1)\frac{B_2}{s^2} + (k+3)(k+2)(k+1)\frac{B_3}{s^3} - \dots$$

Then*

$(viii)$

$$\psi = (1-\mu_1^2) \left[\sum_{n=1}^{n=m} \left\{ A_n \ \frac{(-1)^{n-1}(n-1)!}{r_1^n} \ P'_n \right\} \right.$$

$$\left. - \sum_{k=1}^{\infty} \left\{ \frac{r_1^{k+1}}{s^{k+1}} \cdot \frac{R_{k+1}}{k+1} \ P'_k \right\} \right],$$

where P' is written for $\frac{d}{d\mu_1} \left\{ P(\mu_1) \right\}$.

$$\S \ 2.$$

The equations of the spheroids, with centres at O_1 and O_2, can be written as

$$r_1 = a \left\{ 1 + \epsilon_1 \ P_2(\mu_1) \right\},$$

(ix)

$$r_2 = b \left\{ 1 + \epsilon_2 \ P_2(\mu_2) \right\},$$

where ϵ_1 and ϵ_2 are two small quantities whose second and higher powers are supposed to be negligible; thus

$$r_1^{-n} = a^{-n} \left\{ 1 - n \ \epsilon_1 \ P_2(\mu_1) \right\},$$

$$r_1^{n+1} = a^{n+1} \left\{ 1 + (n+1) \ \epsilon_1 \ P_2(\mu_1) \right\}.$$

Then from (iii^1) and $(viii)$, we get

$$- \tfrac{1}{2} u_1 \ r_1^2 \ (1-\mu_1^2) + \text{constant}$$

(x)

$$= (1-\mu_1{}^2)\left[\ \sum_{n=1}^{n=m}\left\{\ A_n\ \frac{(-1)^{\frac{n-1}{2}}(n-1)!}{r_1{}^n}\ P'_n\ \right\}\right.$$

$$\left. - \sum_{k=1}^{\infty}\left\{\ \frac{r_1{}^{k+1}}{s^{k+1}}\cdot\frac{R_{k+1}}{k+1}\ P'_k\ \right\}\ \right],$$

when $r_1 = a\ \{1 + \epsilon_1\ P_2\ (\mu_1)\}$.

Evidently the constant is zero; we have therefore

$$- \tfrac{1}{2}\ u_1\ a^2\ \{1+2\ \epsilon_1\ P_2\ \}$$

$$= \sum_{n=1}^{n=m}\left[\frac{(-1)^{\frac{n-1}{2}}(n-1)!}{a^n}\ A_n\ \{1-n\epsilon_1\ P_2\}\ P'_n\ \right]$$

$$- \sum_{k=1}^{\infty}\left[\frac{a^{k+1}}{s^{k+1}}\cdot\frac{R_{k+1}}{k+1}\ \{1 + (k+1)\ \epsilon_1\ P_2\}\ P'_k\ \right].$$

Since, by Christoffel's formula*

(ii)

$$P'_n = (2n-1)\ P_{n-1} + (2n-5)\ P_{n-3} + (2n-9)\ P_{n-5} + \ldots,$$

and, consequently,

$$P_2\ P'_n = \frac{3n(n+1)}{2(2n+1)}\ (P_{n+1}-P_{n-1})+(2n-1)\ P_{n-1}+(2n-5)\ P_{n-3}+\ldots,$$

the right-hand member in each equality ending with the term $3\ P_1$, if n is even and with the term P_2, if n is odd. We have

$$- \tfrac{1}{2}\ u_1\ a^2\ \{1 + 2\epsilon_1\ P_2\}$$

$$= \sum_{n=1}^{n=m}\frac{(-1)^{\frac{n-1}{2}}(n-1)!}{a^n}\ A_n\ \left[(2n-1)\ P_{n-1}+(2n-5)\ P_{n-3}+\ \quad\ldots\right]$$

$$- \sum_{n=1}^{n=m}\frac{(-1)^{\frac{n-1}{2}}n!}{a^n}\ \epsilon_1\ A_n\ \left[\frac{3n(n+1)}{2(2n+1)}\ (P_{n+1}-P_{n-1})+(2n-1)\ P_{n-1}+\ldots\right]$$

(iii)

$$- \sum_{k=1}^{\infty}\frac{a^{k+1}}{s^{k+1}}\cdot\frac{R_{k+1}}{k+1}\ \left[(2k-1)\ P_{k-1}+(2k-5)\ P_{k-3}+\ \quad\ldots\right]$$

$$- \sum_{k+1}^{\infty}\frac{a^{k+1}}{s^{k+1}}\cdot\epsilon_1\ R_{k+1}\ \left[\frac{3k(k+1)}{2(2k+1)}\ (P_{k+1}-P_{k-1})+(2k-1)\ P_{k-1}+\ldots\right].$$

* See Heine's "*Kugelfunktionen*," Bd. 1, p. 98.

It remains to equate to zero the co-efficients of the various zonal harmonics.

Equating the co-efficients of P_0.

$$- \tfrac{1}{2} u_1 a^2 = \left[\frac{A_1}{a} + 2! \frac{A_3}{a^3} + 4! \frac{A_5}{a^5} + \qquad \cdots \right]$$

$$- \epsilon_1 \left[3! \frac{A_3}{a^3} + 5! \frac{A_5}{a^5} + 7! \frac{A_7}{a^7} + \qquad \cdots \right]$$

(xiii)

$$- \left[\frac{a^2}{s^2} \cdot \frac{R_2}{2} + \frac{a^4}{s^4} \cdot \frac{R_4}{4} + \frac{a^6}{s^6} \cdot \frac{R_6}{6} + \qquad \cdots \right]$$

$$- \epsilon_1 \left[\frac{a^4}{s^4} R_4 + \frac{a^6}{s^6} R_6 + \frac{a^8}{s^8} R_8 + \qquad \cdots \right].$$

Equating the co-efficients of P_2,

$$- u_1 \epsilon_1 a^2 = 5 \left[2! \frac{A_3}{a^3} + 4! \frac{A_5}{a^5} + 6! \frac{A_7}{a^7} + \cdots \right]$$

$$- 5\epsilon_1 \left[3! \frac{A_3}{a^3} + 5! \frac{A_5}{a^5} + 7! \frac{A_7}{a^7} + \cdots \right]$$

(xiv)

$$- \epsilon_1 \left[\frac{A_1}{a} - 3! \frac{18}{7} \cdot \frac{A_3}{a^3} \right]$$

$$- 5 \left[\frac{a^4}{s^4} \cdot \frac{R_4}{4} + \frac{a^6}{s^6} \cdot \frac{R_6}{6} + \qquad \cdots \right]$$

$$- 5\epsilon_1 \left[\frac{a^4}{s^4} R_4 + \frac{a^6}{s^6} \cdot R_6 + \qquad \cdots \right]$$

$$- \epsilon_1 \left[\frac{a^2}{s^2} R_2 - \tfrac{18}{7} \cdot \frac{a^4}{s^4} R_4 \right].$$

Subtracting (xiii) from (xiv) multiplied by 5, we get

$$-\tfrac{1}{2} u_1 a^2 (5-2\epsilon_1) = \frac{A_1}{a} (5+\epsilon_1) - \tfrac{108}{7} \epsilon_1 \frac{A_3}{a^3}$$

$$- \tfrac{1}{2} \frac{a^2}{s^2} R_2 (5-2\epsilon_1) - \tfrac{18}{7} \cdot \frac{a^4}{s^4} \epsilon_1 R_4$$

To evaluate A_1, we neglect the terms[*] involving A_3, R_2 and R_4. Then, up to the first approximation, we get

$$A_1 = -\tfrac{1}{2} n_1 a^3 \left(\frac{5-2\epsilon_1}{5+\epsilon_1} \right),$$

and consequently by symmetry,

$$B_1 = -\tfrac{1}{2} n_2 b^3 \left(\frac{5-2\epsilon_2}{5+\epsilon_2} \right).$$

It should be noted that, in getting these approximate values, we have neglected terms of the orders higher than $\left(\dfrac{\text{linear dimension}}{\text{central distance}} \right)^2$. Hence to this order of approximation, the velocity potential, for the motion of two spheroids in an infinite liquid, is given by

$$(xv) \quad \phi = -\tfrac{1}{2} n_1 a^3 \left(\frac{5-2\epsilon_1}{5+\epsilon_1} \right) \frac{d}{ds_{12}} \left(\frac{1}{r_1} \right)$$

$$-\tfrac{1}{2} n_2 b^3 \left(\frac{5-2\epsilon_2}{5+\epsilon_2} \right) \frac{d}{ds_{21}} \left(\frac{1}{r_2} \right).$$

If we put each $\epsilon = 0$, so that the spheroids become spheres of radii a and b, we have the velocity potential for the motion of two spheres in an infinite liquid, along their line of centres, to the first order of approximation, as

$$\phi = -\tfrac{1}{2} n_1 a^3 \frac{d}{ds_{12}} \left(\frac{1}{r_1} \right) -\tfrac{1}{2} n_2 b^3 \frac{d}{ds_{21}} \left(\frac{1}{r_2} \right),$$

which is Prof. Karl Pearson's result.[†]

§ 3.

Let us now proceed to approximate still closer to the value of ϕ by retaining the differential coefficients of the second and higher orders. Then from (xii), by equating the coefficients of P_n, we get, for $n > 0$ and $\neq 2$,

$$0 = (-1)^n (2n+1) \left[n! \frac{A_{n+1}}{a^{n+1}} + (n+2)! \frac{A_{n+3}}{a^{n+3}} + \dots \right]$$

$$- (-1)^n (2n+1)\epsilon_1 \left[(n+1)! \frac{A_{n+1}}{a^{n+1}} + (n+3)! \frac{A_{n+3}}{a^{n+3}} + \dots \right]$$

$$+ (-1)^n \frac{A_{n+1}}{a^{n+1}} \epsilon_1 \frac{3(n+1)(n+2)}{2(2n+3)} (n+1)!$$

---

[*] The term involving A_3 is of the same order as $\epsilon_1{}^2$; since A_3 is of the same order as ϵ_1, as will be clear from the value of A_3 obtained on p. 11. Therefore this term is negligible.

[†] Ibid p. 1.

$$- (-1)^{n-2} \frac{A_{n-1}}{a^{n-1}} \, \epsilon_1 \, \frac{3 \, (n-1)n}{2 \, (2n-1)} \, (n-1) \, !$$

$$- (2n+1) \left[\frac{a^{n+2}}{s^{n+2}} \cdot \frac{R_{n+2}}{n+2} + \frac{a^{n+4}}{s^{n+4}} \cdot \frac{R_{n+4}}{n+4} + \, \cdots \, \right]$$

$$- (2n+1) \, \epsilon_1 \left[\frac{a^{n+2}}{s^{n+2}} \cdot R_{n+2} + \frac{a^{n+4}}{s_{n+4}} \, R_{n+4} + \, \cdots \, \right]$$

$$+ \frac{a}{s^{n+2}} \, \epsilon_1 \, R_{n+2} \, \frac{3(n+1)(n+2)}{2(2n+3)}$$

$$- \frac{a^n}{s^n} \, \epsilon_1 \, R_n \, \frac{3(n-1)n}{2(2n-1)}.$$

Equating the coefficients of P_{n+2}.

$$0 = (-1)^{n+2} \, (2n+5) \left[(n+2) \, ! \, \frac{A_{n+3}}{a^{n+3}} + (n+4) \, ! \, \frac{A_{n+5}}{a^{n+5}} + \cdots \right]$$

$$- (-1)^{n+2} \, (2n+5) \, \epsilon_1 \left[(n+3) \, ! \, \frac{A_{n+3}}{a^{n+3}} + (n+5) \, ! \, \frac{A_{n+5}}{a^{n+5}} + \cdots \right]$$

$$+ (-1)^{n+2} \, \frac{A_{n+3}}{a^{n+3}} \, \epsilon_1 \, \frac{3(n+3) \, (n+4)}{2(2n+7)} \, (n+3) \, !$$

$$- (-1)^n \, \frac{A_{n+1}}{a^{n+1}} \, \epsilon_1 \, \frac{3 \, (n+1) \, (n+2)}{2 \, 2n+3)} \, (n+1) \, !$$

$$- (2n+5) \left[\frac{a^{n+4}}{s^{n+4}} \cdot \frac{R_{n+4}}{n+4} + \frac{a^{n+6}}{s^{n+6}} \cdot \frac{R_{n+6}}{n+6} + \, \cdots \, \right]$$

$$- (2n+5) \epsilon_1 \left[\frac{a^{n+4}}{s^{n+4}} \cdot R_{n+4} + \frac{a^{n+6}}{s^{n+6}} \, R_{n+6} + \, \cdots \, \right]$$

$$+ \frac{a^{n+4}}{s^{n+4}} \, \epsilon_1 \, R_{n+4} \cdot \frac{3(n+3)(n+4)}{2(2n+7)}$$

$$- \frac{a^{n+2}}{s^{n+2}} \, \epsilon_1 \, R_{n+2} \, \frac{3(n+1)(n+2)}{2(2n+3)}$$

Hence we get by subtraction

$$0 = (-1)^n n \, ! \, \frac{A_{n+1}}{a^{n+1}} \left[(2n+1) \, (2n+5) - (n^2+3n-1) \, (n+1) \epsilon_1 \right]$$

$$. - (-1)^n \epsilon_1 \; \frac{A_{n-1}}{a^{n-1}} \cdot \frac{3n(n-1)(2n+5)}{2(2n-1)} \; (n-1) \, !$$

$$- (-1)^{n+2} \epsilon_1 \; \frac{A_{n+3}}{a^{n+3}} \; \frac{3(n+3)(n+4)(2n+1)}{2(2n+7)} \; (n+3) \, !$$

$$- \frac{a^{n+2}}{s^{n+2}} \cdot \frac{R_{n+2}}{n+2} \left[(2n+1)(2n+5)+(n^2+3n-1)(n+2)\epsilon_1 \right]$$

$$- \frac{a^n}{s^n} \cdot R_n \epsilon_1 \; \frac{3(n-1)n(2n+5)}{2(2n-1)}$$

$$- \frac{a^{n+1}}{s^{n+4}} \; \epsilon_1 \; R_{n+3} \frac{3(n+3)(n+4)(2n+1)}{2(2n+7)} \, ;$$

of which the terms containing A_{n+3} and R_{n+4} may be neglected as small in comparison with the other terms.[*] Putting $n-2$ for n in this equation, we get

$$0 = (-1)^{n-2} (n-2) \, ! \frac{A_{n-1}}{a^{n-1}} \cdot \left[(2n-3)(2n+1)-(n^2-n-3)(n-1)\epsilon_1 \right]$$

$$- (-1)^{n-2} \epsilon_1 \; \frac{A_{n-3}}{a^{n-3}} \cdot \frac{3(n-3)(n-2)(2n+1)}{2(2n-5)} \; (n-3) \, !$$

$$- \frac{a^n}{s^n} \cdot \frac{R_n}{n} \left[(2n-3)(2n+1) + (n^2-n-3) n \, \epsilon_1 \right]$$

$$- \frac{a^{n-2}}{s^{n-2}} \cdot \epsilon_1 \; R_{n-2} \frac{3(n-3)(n-2)(2n+1)}{2(2n-5)} .$$

Substituting the value of A_{n-1} from this equation in the previous equation, and remembering that ϵ_1^2 is to be neglected, we get

$$0 = (-1)^n n \, ! \; \frac{A_{n+1}}{a^{n+1}} \left[(2n+1)(2n+5)-(n^2+3n-1)(n+1)\epsilon_1 \right]$$

(xvi)
$$- \frac{a^{n+2}}{s^{n+2}} \cdot \frac{R_{n+2}}{n+2} \left[(2n+1)(2n+5)+(n^2+3n-1)(n+2) \epsilon_1 \right]$$

$$- \frac{a^n}{s^n} \cdot R_n \epsilon_1 \cdot \tfrac{1}{2}(n-1)(2n+5)$$

[*] Putting $n+2$ for n in the above equation, we see at once that the term containing A_{n+3} is also of the same order as $\frac{a^{n+4}}{s^{n+4}} \epsilon_1 R_{n+3}$.

Hence from symmetry

$$0 = (-1)^n \, n \, ! \, \frac{B_{n+1}}{b^{n+1}} \left[(2n+1)(2n+5) - (n^2 + 3n - 1)(n+1)\epsilon_2 \right]$$

$$(\textit{xvii}) \qquad - \frac{b^{n+2}}{s^{n+2}} \cdot \frac{S_{n+2}}{n+2} \left[(2n+1)(2n+5) + (n^2 + 3n - 1)(n+2)\epsilon_2 \right]$$

$$- \frac{b^n}{s^n} \cdots S_n \, \epsilon_2 \cdot \frac{3(n-1)(2n+5)}{2}.$$

where

$$S_{k+1} = (k+1) \, \frac{A_1}{s} - (k+2)(k+1) \cdot \frac{A_2}{s^2} + \quad \cdots$$

Thus we have two sets of equations from which by continued approximations we can determine the values of the A's and B's in terms of a, b and s.

<div align="center">§ 4.</div>

Putting $n=4$ in the first equation of § 3, we get

$$0 = 9 \left[4 \, ! \, \frac{A_5}{a^5} + 6 \, ! \, \frac{A_7}{a^7} + \quad \cdots \quad \right]$$

$$- 9 \, \epsilon_1 \left[5 \, ! \, \frac{A_5}{a^5} + 7 \, ! \, \frac{A_7}{a^7} + \quad \cdots \quad \right]$$

$$+ \frac{A_5}{a^5} \quad \epsilon_1 \cdot \frac{3 \cdot 5 \cdot 6}{2 \cdot 11} \quad 5 \, !$$

$$\frac{A_3}{a^3} \quad \epsilon_1 \quad \frac{3 \cdot 3 \cdot 4}{2 \cdot 7}$$

$$- 9 \left[\frac{a^6}{s^6} \cdot \frac{B_6}{6} + \frac{a^8}{s^8} \quad \frac{B_8}{8} + \quad \cdots \quad \right]$$

$$- 9 \epsilon_1 \left[\frac{a^6}{s^6} \cdot B_6 + \frac{a^8}{s^8} \, B_8 + \quad \cdots \quad \right]$$

$$+ \frac{!a^6}{s^6} \, \epsilon_1 \, B_6 \, \frac{3 \cdot 5 \cdot 6}{2 \cdot 11}$$

$$- \frac{a^4}{s^4} \, \epsilon_1 \, B_4 \, \frac{3 \cdot 3 \cdot 4}{2 \cdot 7}$$

From this equation and (*viv*) we get

$$-9n_1\epsilon_1 a^2 = \frac{A_3}{a^3}\ 18(5-3\epsilon_1)-9\epsilon_1\frac{A_1}{a}\ -5\epsilon_1\frac{A_5}{a^5}\cdot\frac{3\cdot5\cdot6}{2\cdot11}\ 5\ !$$

$$-\frac{a^4}{s^4}\cdot\frac{R_4}{4}\ 9(5+4\epsilon_1)-\frac{a^2}{s^2}\ 9\epsilon_1 R_2-5\frac{a^6}{s^6}\ \epsilon_1 R_6\frac{3\cdot5\cdot6}{2\cdot11}$$

Neglecting the terms involving A_5, R_4 and R_6 which are of order higher

than $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)$, we get

$$-9n_1\epsilon_1 a^2 = \frac{A_3}{a^3}\ 18(5-3\epsilon\ |-9\epsilon_1\frac{A_1}{a}\ -9\epsilon_1\frac{a^2}{s^2}\ R_2$$

Substituting the value of A_1, we get for the first approximation

$$A_3 = -\ \frac{3}{5}\ n_1 a^5\left(\frac{\epsilon_1}{5-3\epsilon_1}\right)-\frac{n_2}{2}\cdot\frac{a^5 b^3}{s^3}\left(\frac{5-2\ \epsilon_2}{5+\epsilon_2}\right)\left(\frac{\epsilon_1}{5-3\epsilon_1}\right).$$

We next proceed to approximate to the remaining values of the A's and B's. We see that generally, for $n>o$ and $\neq2$

$$A_{n+1} = \frac{(-1)^n}{n!}\cdot\frac{a^{2n+3}}{s^{n+3}}\cdot\frac{R_{n+2}}{n+2}$$

$$\left[1 + \frac{(n^2+3n-1)(2n+3)\epsilon_1}{(2n+1)(2n+5)-(n^2+3n-1)(n+1)\epsilon_1}\right]$$

$$+ (-1)^n\cdot\frac{a^{2n+1}}{n}\cdot R_n\epsilon_1 \times$$

$$\frac{3(n-1)(2n+5)}{n!\ 2[(2n+1)(2n+5)-(n^2+3n-1)(n+1)\epsilon_1]}\ ,$$

and, consequently, by symmetry

$$B_{n+1} = \frac{(-1)^n}{n!}\cdot\frac{b^{2n+3}}{s^{n+2}}\cdot\frac{S_{n+2}}{n+2}$$

$$\left[1 + \frac{(n^2+3n-1)(2n+3)\epsilon_2}{(2n+1)(2n+5)-(n^2+3n-1)(n+1)\epsilon_2}\right]$$

$$+(-1)^n\frac{b^{2n+1}}{s^n}\cdot S_n\epsilon_2\times\frac{3(n-1)(2n+5)}{n!2[(2n+1)(2n+5)-(n^2+3n-1)(n+1)\epsilon_2]}$$

Examples :—

$$A_2 = \frac{u_2}{2} \cdot \frac{a^5 b^3}{s^4} \left(\frac{5-2\epsilon_2}{5+\epsilon_2}\right) \left\{1 + \frac{5\epsilon_1}{7-2\epsilon_1}\right\},$$

$$A_4 = \frac{u_2}{2 \cdot 3!} \cdot \frac{a^9 b^3}{s^6} \left(\frac{5-2\epsilon_2}{5+\epsilon_2}\right) \left\{1 + \frac{153\epsilon_1}{77-68\epsilon_1}\right\}$$

$$+ \frac{u_2}{2 \cdot 2!} \cdot \frac{a^7 b^3}{s^4} \left(\frac{5-2\epsilon_2}{5+\epsilon_2}\right) \left(\frac{33\epsilon_1}{77-68\epsilon_1}\right),$$

$$B_2 = \frac{u_1}{2} \cdot \frac{b^5 a^3}{s^4} \left(\frac{5-2\epsilon_1}{5+\epsilon_1}\right) \left\{1 + \frac{5\epsilon_2}{7-2\epsilon_2}\right\},$$

$$B_4 = \frac{u_1}{2 \cdot 3!} \cdot \frac{b^9 a^3}{s^6} \left(\frac{5-2\epsilon_1}{5+\epsilon_1}\right) \left\{1 + \frac{153\epsilon_2}{77-68\epsilon_2}\right\}$$

$$- \frac{u_1}{2 \cdot 2!} \cdot \frac{b^7 a^3}{s^4} \left(\frac{5-2\epsilon_1}{5+\epsilon_1}\right) \left(\frac{33\epsilon_2}{77-68\epsilon_2}\right),$$

On the determination of a rough surface on which a moving partile may describe a prescribed path.

By

Nalinimohan Basu.

The object of the present paper is to show how the solution of the following problem can be made to depend on the solution of an *ordinary linear differential equation* :

" To find the *rough* surface on which a moving particle may describe a prescribed curve."

A very simple case of this problem, *viz.*, that in which the surface is *smooth* and gravity the only external force, was studied by the distinguished Belgian-mathematician Catalan.[*]

That the ordinary differential equation is generally not soluble by quadratures should not surprise us, because, as is well known, the motion of a particle on a rough surface has been shown to be determinable by quadratures in only a small number of cases.[†]

I should like to express my indebtedness to Dr. Ganesh Prasad, at whose suggestion I took up, and under whose guidance I carried on, the investigation of this problem.

1. Let us first consider the case when the external force is simply gravity.

Taking the axis of z vertically upwards and unity for the mass of the particle, the equations of motion are

$$\ddot{x} \;=\; l\,R - \mu\,R\,\frac{dx}{ds}\,,$$

$$\ddot{y} \;=\; m\,R - \mu\,R\,\frac{dy}{ds}\,,$$

$$\ddot{z} \;=\; n\,R - g - \mu\,R\,\frac{dz}{ds}\,,$$

where l, m, n are the direction-cosines of the normal to the required surface, μ the co-efficient of friction and R the normal re-action.

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[*] "Sur une probleme de mécanique" (*Journal de Mathématiques*, Series 1, tome 11).

[†] See p. 507 of Prof. Stäckel's article on dynamics in the " Encyclopädie der Mathematischen Wissenschaften " Vol. IV.

Eliminating R we have

$$\frac{\ddot{x}}{l - \mu \frac{dx}{ds}} = \frac{\ddot{y}}{m - \mu \frac{dy}{ds}} = \frac{\ddot{z} + g}{n - \mu \frac{dz}{ds}} = \frac{\frac{d}{dt}\left(\frac{v^2}{2} + gz\right)}{-\mu v} \quad (1)$$

Thus we have

$$\mu v \, \ddot{x} = \left(\mu \frac{dx}{ds} - l\right) \frac{d}{dt} \left(\tfrac{1}{2} v^2 + gz\right),$$

$$\mu v \, \ddot{y} = \left(\mu \frac{dy}{dx} - m\right) \frac{d}{dt} \left(\tfrac{1}{2} v^2 + gz\right).$$

Writing $\ddot{x} = \frac{d}{dt}\left(v \frac{dx}{ds}\right), \ddot{y} = \frac{d}{dt}\left(v \frac{dy}{ds}\right), \frac{d}{dt} = v \frac{d}{ds}$, and

simplifying, we get

$$\mu v^2 \frac{d^2 x}{ds^2} + l \, v \frac{dv}{ds} + \left(l - \mu \frac{dx}{ds}\right) g \frac{dz}{ds} = 0,$$

$$\mu v^2 \frac{d^2 y}{ds^2} + m \, v \frac{dv}{ds} + \left(m - \mu \frac{dy}{ds}\right) g \frac{dz}{ds} = 0.$$

Hence we obtain

$$\frac{\mu v^2}{\mu \left(m \frac{dx}{ds} - l \frac{dy}{ds}\right)} = \frac{v \frac{dv}{ds}}{\left(l - \mu \frac{dx}{ds}\right) \frac{d^2 y}{ds^2} - \left(m - \mu \frac{dy}{ds}\right) \frac{d^2 x}{ds^2}}$$

$$= \frac{g \frac{dz}{ds}}{m \frac{d^2 x}{ds^2} - l \frac{d^2 y}{ds^2}} \qquad \dots \quad (2)$$

$$\therefore v^2 = \frac{g \frac{dz}{ds} \left(m \frac{dx}{ds} - l \frac{dy}{ds}\right)}{m \frac{d^2 x}{ds^2} - l \frac{d^2 y}{ds^2}} \qquad \dots \quad (3)$$

2. Now, let the equations of the given curve be

$$x = f_1(z), \, y = f_2(z),$$

f_1 and f_2 being two known functions.

Then, the equation of the required surface may be written in the form

$$\chi \equiv x - f_1(z) + \{y - f_2(z)\} \phi(x, y, z) = 0 :$$

our problem is to determine the necessary form of ϕ.

Now

$$ds^2 = dx^2 + dy^2 + dz^2 = dz^2 \left\{ 1 + [f'_1(z)]^2 + [f'_2(z)]^2 \right\}$$

$$= \frac{dz^2}{\{\rho(z)\}^2},$$

where $\dfrac{1}{\{\rho(z)\}^2}$ stands for $1 + [f'_1(z)]^2 + [f'_2(z)]^2$.

$$\therefore \frac{dz}{ds} = \rho(z),$$

$$\frac{dx}{ds} = \rho(z) f'_1(z), \quad \frac{dy}{ds} = \rho(z) f'_2(z),$$

$$\frac{d^2x}{ds^2} = [\rho(z)]^2 f_1''(z) + \rho(z) \rho'(z) f_1'(z),$$

$$\frac{d^2y}{ds^2} = [\rho(z)]^2 f_2''(z) + \rho(z) \rho'(z) f_2'(z).$$

Also $\dfrac{\delta\chi}{\delta x} = 1 + \{y - f_2(z)\} \dfrac{\delta\phi}{\delta x}$

$$= 1 \text{ for all points on the curve.}$$

Similarly $\dfrac{\delta\chi}{\delta y} = P$ for all points on the curve,

where $P = \phi\{f_1(z), f_2(z), z\}$;

$$\frac{\delta\chi}{\delta z} = - \{f_1'(z) + P f_2'(z)\}.$$

$$\therefore \frac{l}{-\dfrac{\delta\chi}{\delta x}} = \frac{m}{\dfrac{\delta\chi}{\delta y}} = \frac{n}{\dfrac{\delta\chi}{\delta z}} = \frac{1}{\Omega}.$$

where $\Omega^2 = 1 + P^2 + \{f'_1(z) + P f'_2(z)\}^2$.

$$\therefore l = \frac{1}{\Omega}, \quad m = \frac{P}{\Omega} \text{ and } n = - \frac{\{f'_1(z) + P f'_2(z)\}}{\Omega}$$

Hence from (3) we have

$$v^2 = \frac{g [\rho(z)]^2 \{P f_1'(z) - f_2'(z)\}}{P [\{\rho(z)\}^2 f_1''(z) + \rho(z)\rho'(z)f_1'(z)] - [\{\rho(z)\}^2 f_2''(z) + \rho(z)\rho'(z)f_2'(z)]}$$

$$= \theta(z) \text{ say.}$$

$$\therefore v \frac{dr}{ds} = \tfrac{1}{2} \theta'(z) \quad \frac{dz}{ds} = \tfrac{1}{2} \rho(z) \theta'(z).$$

$\theta'(z)$ containing P and $\dfrac{dP}{dz}$.

If we substitute this value of $r \frac{dv}{ds}$ in the second of the equations (2) we get a differential equation in z from which P can be determined, although not necessarily by quadratures as the differential equation is generally of a complicated form and is insoluble by quadratures except in special cases.

3. Let the value of P thus determined be denoted by $F(z)$. Then one of the forms in which $\phi (x, y, z)$ can be expressed is given by

$$\phi (x, y, z) = F(z) + \{x - f_1(z)\} \psi_1 (x, y, z)$$
$$+ \{y - f_2(z)\} \psi_2 (x, y, z),$$

where ψ_1 and ψ_2 are perfectly arbitrary functions.

Substituting this value of $\phi (x, y, z)$ in the equation of the surface we get a class of equations of the required surface.

4. Let us now consider the case in which the particle moves under *any conservative system of external forces.*

If the components of the external forces are given by $X = \frac{\delta U}{\delta x}$ $Y = \frac{\delta U}{\delta y}$, $Z = \frac{\delta U}{\delta z}$, the equations of motion will be

$$\ddot{x} = X + lR - \mu R \frac{dx}{ds} ; \text{ and two similar equations.}$$

Then proceeding exactly as in Art. 1. we derive the following two equations : —

$$\mu v^2 \frac{d^2 x}{ds^2} + lv \frac{dv}{ds} - \left(l - \mu \frac{dx}{ds} \right) \frac{dU}{ds} - \mu \frac{\delta U}{\delta x} = 0,$$

$$\mu v^2 \frac{d^2 y}{ds^2} + mv \frac{dv}{ds} - \left(m - \mu \frac{dy}{ds} \right) \frac{dU}{ds} - \mu \frac{\delta U}{\delta y} = 0,$$

whence we obtain

$$\frac{\mu \left[\left(l \frac{\delta U}{\delta y} - m \frac{\delta U}{\delta x} \right) - \left(l \frac{dy}{ds} - m \frac{dx}{ds} \right) \frac{dU}{ds} \right]}{v \frac{dv}{ds}}$$

$$= \frac{\mu v^2}{\frac{d^2 y}{ds^2} \left[\left(l - \mu \frac{dx}{ds} \right) \frac{dU}{ds} + \mu \frac{\delta U}{\delta x} \right] - \frac{d^2 x}{ds^2} \left[\left(m - \mu \frac{dy}{ds} \right) \frac{dU}{ds} + \mu \frac{\delta U}{\delta y} \right]}$$

$$= \frac{-1}{m \frac{d^2 x}{ds^2} - l \frac{d^2 y}{ds^2}}$$

But, on the curve. $\frac{dU}{ds}$. $\frac{\delta U}{\delta x}$. $\frac{\delta U}{\delta y}$ can be expressed in terms of z

alone. Therefore, as in the previous case, we get a differential equation in z for P, from which P can be determined. If $F(z)$ be the solution of this equation, the equation of the required surface will be of the same form as in Art. 3.

5. *Illustrative examples.* (I) Let the prescribed path be the *circle*

$$x = z \tan a. \quad x^2 + y^2 + z^2 = 1,$$

and let gravity be the only external force. Then, it is easily seen from Art. 2 that the ordinary differential equation for determining the required surface is

$$\frac{d}{dz}\left(\frac{P \sin 2a}{1 - 2 Pz \tan a} \right) = -3 + \frac{\Omega \mu \sin 2a}{(1 - 2 Pz \tan a) \sqrt{\cos^2 a - z^2}},$$

where

$$\Omega^2 \cos^2 a = 1 - 4 Pz \tan a + 4 z^2 P^2 \sec^2 a + 4 P^2 (\cos^2 a - z^2).$$

(II). Let the prescribed path be the *helix*

$$x = \cos kz. \quad y = \sin kz.$$

under the action of gravity.

Then the ordinary differential equation for P is

$$\frac{d}{dz}\left(\frac{P \sin kz + \cos kz}{P \cos kz - \sin kz} \right) = 2k \left(\frac{k c \mu \Omega}{P \cos kz - \sin kz} - 1 \right),$$

where $c^2 = \frac{1}{1 + k^2}$ and $\Omega^2 = 1 + P^2 + k^2 (P \cos kz - \sin kz)^2.$

If we put

$$\frac{P \sin kz + \cos kz}{P \cos kz - \sin kz} = Q.$$

the above equation becomes

$$\frac{dQ}{dz} = -2k + 2k^2 c \mu \sqrt{Q^2 + \frac{1}{c^2}}$$

which is soluble by quadratures.
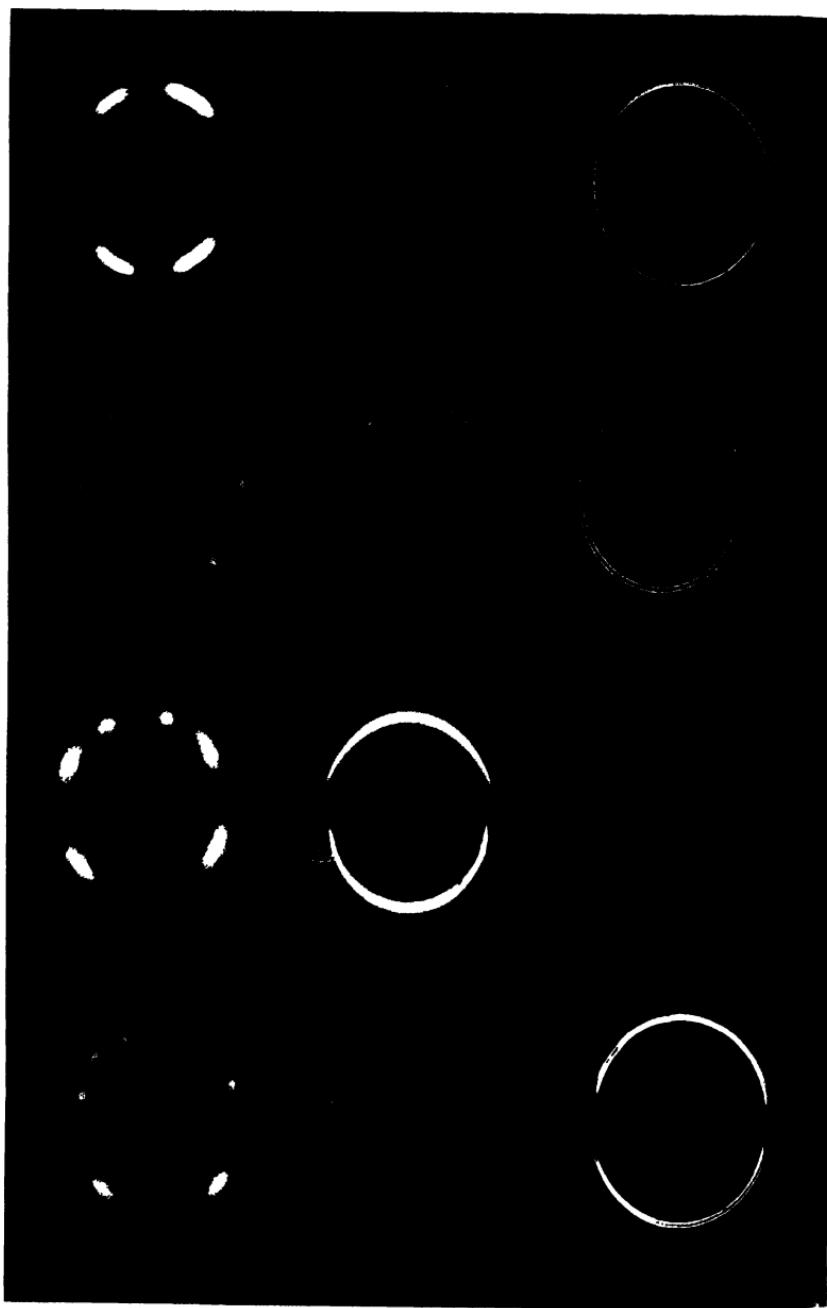
Illustrating the Emission of Light by the Boundary of a Circular
Diffracting Aperture.

Figs

17, 18, 19

20, 21, 22

23, 24, 25
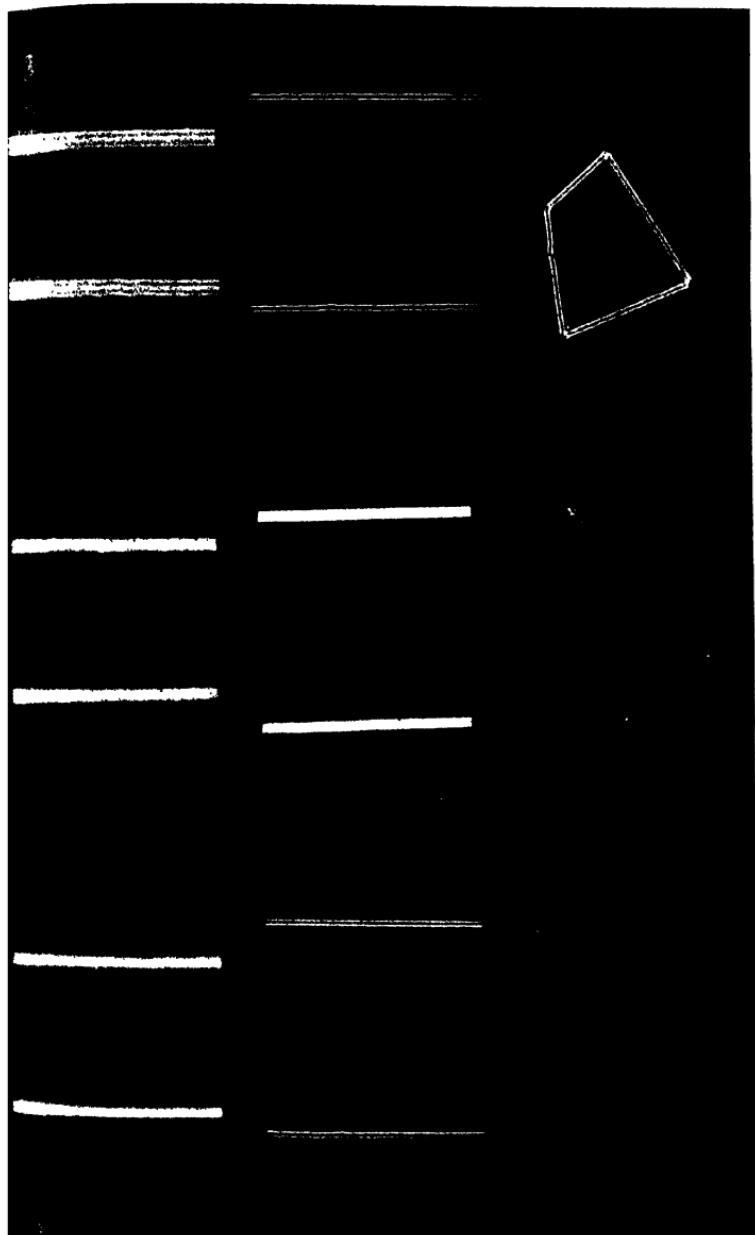
Illustrating the Emission of Light by the Edges of Apertures with
Rectilinear Boundaries.

On the Theory of Foucault's Test and the Radiation from the Edges of Diffracting Apertures.[1]

By

Sudhansukumar Banerji.

1. *Introduction.*

In a recent paper published in the Philosophical Magazine[2], Lord Rayleigh has given a treatment on the principles of the wave theory of the phenomena observed when an optical surface is examined by the well-known 'knife-edge' test due to Foucault in which a screen is placed in the focal plane and advanced gradually so as to cut off the rays coming to a focus, the optical imperfections, if any, becoming evident when the surface is viewed by the eye (or through a telescope) placed behind the focal plane. Assuming that the surface is bounded by parallel straight edges, and that the source of light is also linear, Lord Rayleigh has shown how the luminosity of the boundaries which is observed when the bulk of the light is cut off, may be accounted for as due to the diffraction of the rays reaching the focal plane. He has also considered the effects produced by introducing optical retardations of a linear type over the surface of the aperture. In the present paper, I have extended Lord Rayleigh's method of investigaticn to other cases of interest, particularly those in which the surface under test has a circular boundary and is illuminated by a point source of light. (This, it may be remarked, is the case that most frequently occurs in practice[3]). In the course of the work, it was found that the configuration of the luminous fringes observed at the boundary of the surface presents some very remarkable features which will also be dealt with in this paper. To illustrate the results

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[1] A preliminary note describing the results contained in Section 5 of this paper was published in 'Nature', May 10, 1917.

[2] Lord Rayleigh, "On Methods for detecting small Optical Retardations, and on the Theory of Foucault's Test", Phil. Mag., February, 1917.

[3] See the Memoirs by Draper and Ritchey on the Construction of a Silvered Glass Telescope, Smithsonian Contributions to Knowledge, Vol. XXXIV (1904).

obtained, photographic records have been secured, a selection from
which is reproduced in the Plate.   These phenomena have an
important bearing on the general theory of diffraction, being evidently
related to the well-known Gouy-Sommerfeld effect of the emission
of light by a diffracting edge[1].   I have also attempted to consider
the effects noticed in Foucault's test, when the retardations are of a
nonlinear type, that is when the foci of the pencils do not all lie in
the plane of the knife-edge.   In the practical applications of
Foucault's test, these are probably more common than retardations of
the linear type.

2.   Case of the Circular Boundary.

As remarked above, the optical surfaces examined by the 'knife-
edge' test are most frequently limited by a circular aperture, the
illumination being that due to a point source.   Fig. 9 (Plate)
reproduces a photograph of the luminosity observed at the boundary
when the knife-edge is put in horizontally into the focal plane from
below so as to cut off most of the light.   It will be noticed that the
luminosity is a maximum on the upper and the lower boundaries, and
diminishes to zero at the ends of a horizontal diameter.   In order to
give definiteness to a discussion of this effect, it is necessary to
postulate some specified form for the boundaries of the aperture in
the focal plane which admits the diffracted rays into the field of view
of the observing telescope.   For instance, we may assume that a
horizontal slit is placed in the focal plane below the centre of the field.
Fig. 12 in the Plate reproduces the beautiful lunette-shaped diffraction
fringes that appear on either side of the boundary of the circular
aperture with this arrangement.   The luminosity, as in the case of
the simple knife-edge test, tends to zero at the ends of a horizontal
diameter.   The explanation of this fact and of the peculiar form of
the fringes will appear later.

More striking still are the interference phenomena obtained when
the boundary is observed through a pair of apertures of the same form
placed in the focal plane.   With two horizontal slits placed *on the same*

[1]   Gouy, Ann. d. Phys. et de Chim., (6), 8, p. 145, (1886).   Sommerfeld, Math.
Ann., Vol. XLVII, p. 317, (1896).   See also, E. Macy, Wiedemann's Annalen, 49,
(1893), and Kalaschnikow, Journ. Russ. Phys. Ges. 44 (1912).   For a rigorous
treatment of the problem of diffraction of light by a rectilinear slit, see Schwarzs-
child, Math. A: n.. Vol. 55, (1902).

side of the centre of the field, lunette-shaped *interference* fringes are observed, the c·ntral fringe which coincides with the boundary being white (fig. 15 in the Plate). But when two horizontal slits are placed *on opposite sides* of the centre of the field, the *central fringe is black*, in other words, the boundary of the aperture is itself non-luminous but appears surrounded on either side by luminous bands (fig. 6 in the Plate). This remarkable fact is one of great generality. In all cases in which the apertures in the focal plane through which the diffracted rays pass (whatever be their actual form) are symmetrically disposed about the centre of the field, the latter itself being excluded, the image of the boundary of the diffracting surface appears as a black line. This is irrespective of the actual form of the boundary itself, that is, whether it is circular or of any other shape whatsoever. Fig. 16 in the Plate represents the appearance of the circular boundary when a horizontal wire is placed across the centre of the focal plane. A fine black line may be seen running through the luminous arcs and dividing them into two. A case that admits of detailed mathematical treatment is that in which the arrangement is completely symmetrical about the axis. Figs. 7 and 10 in the Plate show the results obtained when the central part of the field at the focal plane is blocked out by a circular disk and only the diffracted rays, passing through an annulus of greater or less width surrounding it, enter the observing telescope. It will be seen that in both photographs the boundary appears as a perfectly black circle, with luminous rings on either side of it.

Let R be the radius of the circular aperture of the lens and assume that in the focal plane, there is a screen containing an annular aperture, R_1, R_2 being the radii of the circles defining the annulus. Let ϕ be the angle of diffraction of parallel rays which meet at any point Q in the focal plane. Since the path-difference between the rays leaving a point (r, θ) and the centre of the diffracting aperture is evidently $r \cos \theta \sin \phi$, the diffracted disturbance at a point in the focal plane due to an area $r\,dr\,d\theta$ can be written in the form

$$r \sin 2\pi \left(\frac{t}{T} - \frac{r \cos \theta \sin \phi}{\lambda} \right) dr\,d\theta.$$

The total disturbance at Q in the focal plane is therefore

$$\int_0^{2\pi} \int_0^R r \sin 2\pi \left(\frac{t}{T} - \frac{r \cos \theta \sin \phi}{\lambda} \right) dr\,d\theta. \tag{1}$$

The rays from the various elements $r'dr'd\theta'$ of the second aperture may be regarded as meeting in the field of the observing telescope proceeding at an angle ϕ' with the axis and producing the observed effect. The total disturbance in this direction due to the second aperture is therefore

$$\int_0^{2\pi}\int_{R_1}^{R_2}\int_0^{2\pi}\int_0^{R} r' \sin 2\pi \left(\frac{t}{T} - \frac{r\cos\theta\sin\phi}{\lambda}\right.$$
$$\left. - \frac{r'\cos\theta'\sin\phi'}{\lambda}\right) dr\,d\theta\,dr'\,d\theta'. \qquad (2)$$

Since ϕ and ϕ' are small quantities the above expression can be written as

$$\int_0^{2\pi}\int_{R_1}^{R_2}\int_0^{2\pi}\int_0^{R} rr' \sin 2\pi \left(\frac{t}{T} - \frac{\phi.\,r\cos\theta}{\lambda} - \frac{\phi'.\,r'\cos\theta'}{\lambda}\right) dr\,d\theta\,dr'\,d\theta'. \qquad (3)$$

This expression can be written in the form

$$\int_0^{2\pi}\int_{R_1}^{R_2} r'\sin 2\pi \left(\frac{t}{T} - \frac{\phi'.\,r'\cos\theta'}{\lambda}\right) dr'\,d\theta' \int_0^{2\pi}\int_0^{R}$$
$$r\cos\left(\frac{2\pi\phi.\,r\cos\theta}{\lambda}\right) dr\,d\theta, \qquad (4)$$

the other integral being zero on account of symmetry of the diffracting aperture.

But the integral

$$\int_0^{2\pi}\int_0^{R} r\cos\left(2\pi\frac{\phi.\,r\cos\theta}{\lambda}\right) dr\,d\theta = \frac{2}{\pi}.\frac{R\lambda}{2\pi\phi} J_1\left(\frac{2\pi}{\lambda}R\phi\right)$$

Therefore the expression (4) becomes

$$\int_0^{2\pi}\int_{R_1}^{R_2} r'\,d\theta'\,dr' \sin 2\pi \left(\frac{t}{T} - \frac{\phi'.\,r'\cos\theta'}{\lambda}\right).\frac{1}{\phi} J_1\left(\frac{2\pi}{\lambda}R\phi\right),$$

neglecting a constant factor. $\qquad (5)$

If f is the focal length of the lens, then $\phi = \dfrac{f'}{f}$.

The expression (5) can be written in the form

$$\sin 2\pi \frac{t}{T} \int_0^{2\pi} \int_{R_1}^{R_2} r'\, d\theta'\, dr' \cos\left(2\pi \frac{\phi'.\, r'\cos\theta'}{\lambda}\right) \frac{f}{r'} J_1\left(\frac{2\pi R}{\lambda f}.\, r'\right)$$

$$- \cos 2\pi \frac{t}{T} \int_0^{2\pi} \int_{R_1}^{R_2} r'\, d\theta'\, dr' \sin\left(2\pi \frac{\phi'.\, r'\cos\theta'}{\lambda}\right) \frac{f}{r'} J_1\left(\frac{2\pi R}{\lambda f}.\, r'\right)$$

Since the second aperture is also symmetrical about the axis, the second integral is zero, for the elements of it arising from two points situated at equal distances on opposite sides of a diameter are equal and of opposite signs. Therefore the intensity as viewed in the direction ϕ' is

$$I = \left[\int_0^{2\pi} \int_{R_1}^{R_2} d\theta'\, dr' \cos\left(2\pi \frac{\phi'.r'\cos\theta'}{\lambda}\right) J_1\left(\frac{2\pi R}{\lambda f}.r'\right)\right]^2$$

Integrating with respect to θ', we get

$$I = \left[\int_{R_1}^{R_2} J_0\left(\frac{2\pi\phi'}{\lambda} r'\right) J_1\left(\frac{2\pi R}{\lambda f}.r'\right) dr'\right]^2;$$

(neglecting a constant factor).

If the angular semi-aperture of the lens be denoted by ψ, then $R = f\psi$. Therefore

$$I = \left[\int_{R_1}^{R_2} J_0\left(\frac{2\pi\phi'}{\lambda} r'\right) J_1\left(\frac{2\pi\psi}{\lambda} r'\right) dr'\right]^2$$

Since $\dfrac{1}{\lambda}$ is a very large quantity, it is convenient to use semi-convergent expansions for J_0 and J_1. We have

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x - \frac{\pi}{4}\right)\left\{1 - \frac{1^2 \cdot 3^2}{2!(8x)^2} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{4!(8x)^4} - \cdots\right\}\right.$$

$$\left. + \sin\left(x - \frac{\pi}{4}\right)\left\{\frac{1}{8x} - \frac{1^2 \cdot 3^2 \cdot 5^2}{3!(8x)^3} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2}{5!(8x)^5} - \cdots\right\}\right],$$

$$J_1(x) = \sqrt{\frac{2}{\pi x}}\Bigg[\sin\left(x-\frac{\pi}{4}\right)\left\{1+\frac{3\cdot5\cdot1}{8\cdot16}\left(\frac{1}{x}\right)^2\right.$$

$$\left. -\frac{3\cdot5\cdot7\cdot9\cdot1\cdot3\cdot5}{8\cdot16\cdot24\cdot32}\left(\frac{1}{x}\right)^4+\dots\right\}$$

$$+\cos\left(x-\frac{\pi}{4}\right)\left\{\frac{3}{8}\cdot\frac{1}{x}-\frac{3\cdot5\cdot7\cdot1\cdot3}{8\cdot16\cdot24}\left(\frac{1}{x}\right)^3\right.$$

$$\left. +\frac{3\cdot5\cdot7\cdot9\cdot11\cdot1\cdot3\cdot5\cdot7}{8\cdot16\cdot24\cdot32\cdot40}\left(\frac{1}{x}\right)^5-\dots\right\}\Bigg].$$

Thus when x is large,

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x-\frac{\pi}{4}\right)+\frac{1}{8}\;\frac{\sin\left(x-\frac{\pi}{4}\right)}{x}\right],$$

$$J_1(x) = \sqrt{\frac{2}{\pi x}}\left[\sin\left(x-\frac{\pi}{4}\right)+\frac{3}{8}\;\frac{\cos\left(x-\frac{\pi}{4}\right)}{x}\right].$$

Therefore,

$$I = \left[\int_{R_1}^{R_2} J_0\left(\frac{2\pi\phi'}{\lambda}x\right) J_1\left(\frac{2\pi\psi}{\lambda}x\right) dx\right]^2$$

$$= \frac{1}{\psi\phi'}\Bigg[\frac{\lambda}{2\pi^2}\int_{R_1}^{R_2}\frac{\sin\frac{2\pi}{\lambda}(\psi-\phi')x}{x}\,dx$$

$$-\frac{\lambda}{2\pi^2}\int_{R_1}^{R_2}\frac{\cos\frac{2\pi}{\lambda}(\psi+\phi')x}{x}\,dx$$

$$+\frac{\lambda^2}{32\pi^3}\left(\frac{1}{\phi'}+\frac{3}{\psi}\right)\int_{R_1}^{R_2}\frac{\cos\frac{2\pi}{\lambda}(\psi-\phi')x}{x}\,dx$$

$$-\frac{\lambda^2}{32\pi^3}\left(\frac{1}{\phi'}-\frac{3}{\psi}\right)\int_{R_1}^{R_2}\frac{\sin\frac{2\pi}{\lambda}(\psi+\phi')x}{x}\,dx$$

$$+\text{ terms involving higher powers of }\Bigg]^2.$$

Taking $R_1 = \dfrac{3\lambda}{2\pi\psi}$, $R_2 = \dfrac{50\lambda}{2\pi\psi}$, we obtain (neglecting a constant factor),

$$I = \frac{1}{\phi'\psi}\left[\left\{ \text{Si } 50\left(1-\frac{\phi'}{\psi}\right) - \text{Si } 3\left(1-\frac{\phi'}{\psi}\right)\right\}\right.$$
$$- \left\{ \text{Ci } 50\left(1+\frac{\phi'}{\psi}\right) - \text{Ci } 3\left(1+\frac{\phi'}{\psi}\right)\right\}$$
$$\left. + \lambda\left(\ldots \quad \ldots \right) + \ldots \right]^2$$

Calculating the values of the expression for different values of $\dfrac{\phi'}{\psi}$, we construct the following table :—

TABLE I.

$\phi'\psi$	$\sqrt{I}$	I ($\times$ const. factor.)	$\phi'\psi$	$\sqrt{I}$	I ($\times$ const. factor.)
0·00	−·1618	26,179	1·02	−·8891	790,501
0·20	−·2187	47,829	1·04	−1·5266	2,330,657
0·40	−·1190	14,161	1·06	−1·7009	2,891,056
0·60	+·2550	65,025	1·08	−1·5498	2,390,862
0·80	+·8192	671,085	1·10	−1·2687	1,609,597
0·90	+1·1385	1,290,220	1·20	−1·0471	1,096,438
0·92	+1·4169	2,007,612	1·40	−·3597	129,385
0·94	+1·6617	2,761,281	1·60	+·0582	3,387
0·96	+1·4055	1,975,447	1·80	+·2539	64,465
0·98	+·8154	664,878	2·00	+·3579	128,095
0·99	+·3964	157,134	2·10	+·1010	10,201
1·00	+·0732	5,358	2·60	−·0536	2,873
1·01	−·5241	274,581			

Plotting the values, we obtain a curve (fig. 1) representing the distribution of intensity along any given diameter. The fringes that appear on either side of the boundary are clearly shown, the most remarkable feature being the extreme rapidity with which the intensity falls practically to zero on the boundary itself (BB in the figure) from a large value on either side of it. This feature depends on the inner radius R_1 of the annulus being small and the outer radius R_2 being very large, and is entirely confirmed by observation under these conditions.

When the radii R_1 and R_2 of the annulus in the focal plane do not differ much or if they are both large, the brightness falls off to zero on the boundary but not very suddenly. A larger number of well-defined fringes also appear on either side of the boundary in the former case. [See for instance, fig. 13 in the Plate]. This will be shown from the following calculations based on the data obtained from an actual experiment. The data were

$$R = \cdot04 \text{ mm.}, f = 272 \text{ mm.}, R_1 = 1\cdot72 \text{ mm.},$$

$$R_2 = 2\cdot61 \text{ mm.}, \lambda = \cdot00045 \text{ mm.}$$

We thus obtain for this case

$$1 = \frac{1}{\phi'\psi}\Big[\Big\{ \text{Si } 122\cdot67 \Big(1-\frac{\phi'}{\psi}\Big) - \text{Si } 80\cdot84 \Big(1-\frac{\phi'}{\psi}\Big)\Big\}$$

$$-\Big\{ \text{Ci } 122\cdot67 \Big(1+\frac{\phi'}{\psi}\Big) - \text{Ci } 80\cdot84 \Big(1+\frac{\phi'}{\psi}\Big)\Big\}$$

$$+\gamma\Big(\text{.........}\Big) + \text{etc.}\Big]^2.$$

The values of this expression for different values of $\frac{\phi'}{\psi}$ are shown in Table II in which the calculated and observed values of the ratio ϕ'/ψ at which the illumination is a maximum or minimum are given for comparison. It will be seen from the Table that $\sqrt{1}$ changes sign at the minima and these are therefore absolute zeroes.

Fig. 1.

TABLE II.

| ϕ'/ψ	$\sqrt{i}$	I (× const. factor.)	Calculated values of ϕ'/ψ for Max. or Min.	Observed values of ϕ'/ψ for Max. or Min.	ϕ'/ψ	$\sqrt{i}$	I (× const. factor.)	Calculated values of ϕ'/ψ for Max. or Min.	Observed values of ϕ'/ψ for Max. or Min.
0·880	−·0536	2,873	0·885 Min.	…	1·000	+·0053	28	1·000 Min.	1·000
0·890	−·1257	15,801	0·892 Max.	0·894	1·005	−·2134	45,539	…	…
0·900	−·0804	6,464	0·904 Min.	0·901	1·010	−·3448	118,889	…	…
0·910	+·0908	8,245	…	…	1·015	−·4158	172,889	1·017 Max.	1·015
0·920	+·2179	47,481	0·922 Max.	0·930	1·020	−·3673	134,908	…	…
0·930	+·1543	23,808	…	…	1·025	−·2209	48,797	…	…
0·940	−·0779	6,068	0·935 Min.	0·940	1·030	−·0491	2,411	1·028 Min.	1·030
0·950	−·3149	99,168	0·953 Max.	0·957	1·040	+·2866	82,139	…	…
0·960	−·2761	76,231	…	…	1·050	+·3251	105,690	1·047 Max.	1·046
0·970	+·0599	3,588	0·971 Min.	0·971	1·060	+·0875	7,656	1·063 Min.	1·061
0·975	+·2423	58,709	…	…	1·070	−·1478	21,844	…	…
0·980	+·3779	142,809	…	…	1·080	−·2141	45,839	1·077 Max.	1·078
0·985	+·4268	182,158	0·983 Max.	0·985	1·090	−·0859	7,379	…	…
0·990	+·3548	125,883	…	…	1·100	+·0854	7,298	1·095 Min.	1·096
0·995	+·2344	50,355	…	…	1·110	+·1354	18,333	1·107 Max.	1·109
1·000	+·0053	28	1·000 Min.	1·000	1·120	+·0596	3,552	1·125 Min.	1·132
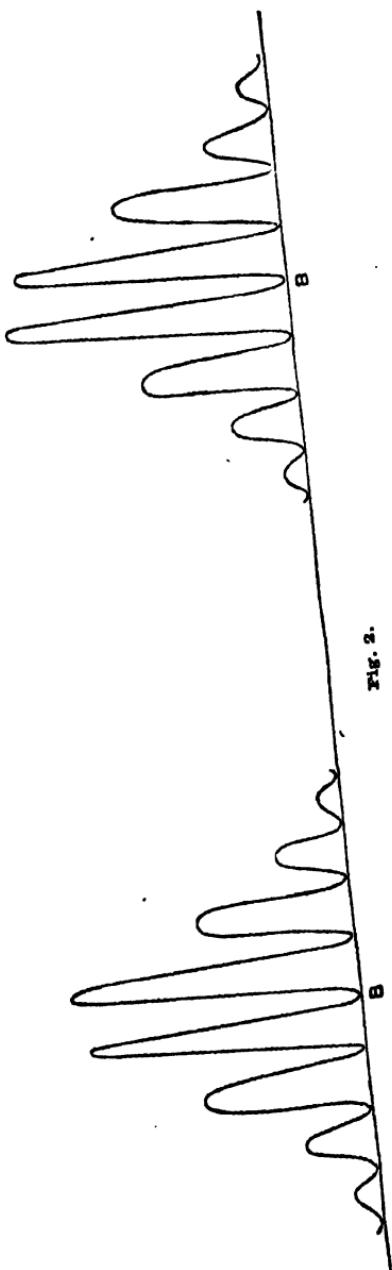
Fig. 2.

The values for I shown in Table II have been plotted in fig. 2.

The investigation given above may be modified to suit the case in which we have one or more slits (instead of an annulus) in the focal plane, by suitably altering the limits of integration in the expression (5) given above. The writer hopes to give the detailed numerical calculations in a later paper. It is possible, however, to understand in a general way the reason for the peculiar configuration of the fringes for these cases shown in figs. 6, 12 and 15 in the Plate. If instead of a slit we have in the focal plane a small aperture at (r',θ') through which the boundary is viewed, the luminosity of the latter appears confined to certain regions lying in the neighbourhood of the two points. $r=R$, $\theta=\theta'$ and $\theta=\pi+\theta'$ and which are more or less well-defined according to the size and position of the aperture. The farther the aperture is from the centre of the focal plane, the feebler is the luminosity observed through it. Accordingly, if we regard the horizontal slit placed in the focal plane as consisting of a number of elements along its length, the vanishing of the luminosity of the boundary at the ends of the horizontal diameter is seen to follow as a consequence. At and near these points also, the radial width of the luminosity should obviously be the least as the latter is, roughly speaking, in inverse proportion to the corresponding radial width of the aperture in the focal plane. This gives us a qualitative explanation of the lunette-shaped form of the fringes in these cases.

Figs. 5, 8, 11 and 14 in the Plate illustrate the remarks made above regarding the localisation of the luminosity of the boundary observed in certain cases. Fig. 5 represents the effect observed when there were two small circular apertures in the focal plane not lying on the same radius vector. Accordingly, we have on the boundary four separate regions of luminosity. Fig. 8 represents a photograph obtained when a ring of six circular holes was placed symmetrically in the focal plane. Each of the six spots seen along the boundary is crossed by very fine fringes due to the interference of the effects produced by the pair of apertures at the end of each diameter. Fig. 14 was obtained when the ring of holes was slightly displaced in the focal plane. Twelve spots appear on the boundary. Fig. 11 represents the effect observed when the screen in the focal plane was so placed that two out of the three pairs of apertures fell on lines passing through the centre of the field. Accordingly, only eight spots are seen, the four larger ones being crossed by fine interference fringes.

13. Case of the Rectangular Boundary.

Some beautiful and stiking colour effects are noticed in the 'knife-edge' test when the surface under observation is bounded by parallel straight edges. (The source of light should be either a point or a fine slit parallel to the boundaries of the surface and to the knife-edge, so that the diffraction fringes formed in the focal plane are sharply defined). When the knife-edge is put in the focal plane so as to cut off all the diffraction bands on one side and also a few of those on the other, it is observed that the boundaries of the surface appear luminous and white, but the region inside the boundaries shows colour, this being practically of the same tint throughout but most marked midway between the boundaries. The whole of the field between the boundaries changes colour as the knife-edge is moved so as to cut off more bands of the diffraction pattern formed at the focus. The colour is successively the following as the knife-edge is moved into the focal plane :—brilliant white, yellowish white, yellow, dark yellow, orange-red, indigo, blue, green, greenish yellow, yellow, orange-yellow, orange, orange-red, purplish red, purple, indigo-blue, bluish green, grass-green, greenish-yellow, orange-yellow, purple, greenish blue, green and so on. The field outside the boundaries also shows a colour (though much less vividly) which is in general complementary to that observed between the boundaries. When the source of light is monochromatic, it is found that the luminosity of the entire field of observation between the boundaries undergoes large fluctuations as the knife-edge is moved in the focal plane. The colour effects noticed in white light are obviously due to these fluctuations of intensity not being in the same phase for different parts of the spectrum.

The theory of Foucault's test as developed by Lord Rayleigh is found to be capable of explaining these remarkable colour phenomena. The expression given by Lord Rayleigh for the intensity of the field as viewed in the direction ϕ is

$$1 = \left[\mathrm{Si} \left\{ \frac{2\pi}{\lambda} \theta \left(1 + \frac{\phi}{\theta} \right) \xi_2 \right\} - \mathrm{Si} \left\{ \frac{2\pi}{\lambda} \theta \left(1 + \frac{\phi}{\theta} \right) \xi_1 \right\} \right.$$

$$\left. + \mathrm{Si} \left\{ \frac{2\pi}{\lambda} \theta \left(1 - \frac{\phi}{\theta} \right) \xi_2 \right\} - \mathrm{Si} \left\{ \frac{2\pi}{\lambda} \theta \left(1 - \frac{\phi}{\theta} \right) \xi_1 \right\} \right]^2$$

$$+ \left[\operatorname{Ci} \left\{ \frac{2\pi}{\lambda}\theta \left(1 - \frac{\phi}{\theta}\right)\xi_2 \right\} - \operatorname{Ci} \left\{ \frac{2\pi}{\lambda}\theta \left(1 - \frac{\phi}{\theta}\right)\xi_1 \right\} \right.$$

$$\left. - \operatorname{Ci} \left\{ \frac{2\pi}{\lambda}\theta \left(1 + \frac{\phi}{\theta}\right)\xi_2 \right\} + \operatorname{Ci} \left\{ \frac{2\pi}{\lambda}\theta \left(1 + \frac{\phi}{\theta}\right)\xi_1 \right\} \right]^2 .$$

where ξ_1, ξ_2 define the aperture in the focal plane and θ is the angular semi-aperture of the lens. In practice ξ_2 is large. To illustrate the fact that the luminosity of the field between the boundaries undergoes fluctuations as ξ_1 is gradually increased, I have prepared the following table showing the intensity of different points of the field as calculated from the expression given above taking $\xi_2 = \frac{300\lambda}{2\pi\theta}$ and $\xi_1 = \frac{3\lambda}{2\pi\theta}x$,

where x has the values 1, 1·63, 2, 2·62 and 3 in succession, these being approximately the values for which the intensity at the centre of the field is greatest or least.

It will be seen from the figures in Table III that the intensity of the entire field between the limits $\phi/\theta = \pm 1$ becomes alternately greater and less as ξ_1 is increased. (This fact has not been brought out in Lord Rayleigh's paper). The region outside the boundaries does not show such a marked variation. In fact, observation shows that when ξ_2 is large, a variation of ξ_1 produces a relatively insignificant effect on the intensity of the field outside the boundaries. If, however, ξ_2 be not large, experiment and theory agree in showing that a variation of either ξ_1 alone or of ξ_1 and ξ_2 ($\xi_2 - \xi_1$ remaining constant) results in a marked fluctuation of the intensity of the field outside the boundaries, and the colours observed in this region with white light become more prominent. It should be remarked here that when ξ_2 is not large, the regions inside and outside the boundaries (observed in white light) are not generally of uniform tint throughout, as diffraction frings appear showing a regular succession of colours, although a preponderance of a particular colour within the boundaries and of a complementary colour in the region outside is noticed. As the knife-edge is moved in, the relative intensities of the different colours change to an enormous extent, and the positions of the different fringes shift to and fro. The changes in the position and the colour of the bands within the boundaries are most interesting to watch when their number is small. If it is arranged to alter both ξ_1 and ξ_2 ($\xi_2 - \xi_1$ remaining constant and small), a fluctuation in the number of the fringes between the

TABLE III.

ζ	$\frac{\theta}{\phi}=\mp 0.$	$\frac{\theta}{\phi}=\mp .1.$	$\frac{\theta}{\phi}=\mp .2.$	$\frac{\theta}{\phi}=\mp .3.$	$\frac{\theta}{\phi}=\mp .5.$	$\frac{\theta}{\phi}=\mp .8.$	$\frac{\theta}{\phi}=\mp .9.$	$\frac{\theta}{\phi}=\mp 1.$	$\frac{\theta}{\phi}=\mp 1.1.$	$\frac{\theta}{\phi}=\mp 1.2.$	$\frac{\theta}{\phi}=\mp 1.5.$	$\frac{\theta}{\phi}=\mp 1.8.$	$\frac{\theta}{\phi}=\mp 2.$
1	·31	·34	·320	·350	·4718	1·0150	1·310	20·23	1·6850	·7550	·1742	·0712	·0384
1·68	0	·0001	·0052	·020	·0625	·2996	1·330	16·82	1·6008	·6205	·1969	·0872	·0601
2	·09	·07	·0830	·095	·1470	·3945	1·009	15·30	·8369	·3277	·0519	·0196	·0349
2·62	0	·0009	·0025	·007	·0058	·1530	·604	18·75	·7270	·2350	·0418	·0132	·0151
3	·01	·033	·0315	·038	·0186	·1782	·544	12·35	·5443	·1781	·0401	·0316	·0120

boundaries from one to two, or two to three, and *vice versa*, may be observed, these changes synchronising with the fluctuation in the intensity of the bands.

Fig. 23 in the Plate represents the luminosity observed at the edges of a rectangular diffracting aperture in Foucault's test. In taking this photograph, ξ_1 was small and ξ_2 large. The luminosity accordingly appears highly condensed at the edges. Fig. 20 reproduces a photograph obtained when ξ_1 and ξ_2 did not differ very considerably. Diffraction fringes are clearly seen on either side of the boundaries in this case. Fig. 17 reproduces a photograph of the aperture obtained with two parallel slits in the focal plane on the same side. It will be noticed that the central fringe which coincides with each boundary is white. Figs. 21 and 24 represent photographs obtained when the central band and a few fringes on either side of the diffraction pattern at the focal plane were cut off by a wire parallel to the edges of the aperture. It will be observed that the positions of the boundaries in these two photographs appear as fine black lines with luminous bands on either side. The same feature but with the dark lines at the boundaries much broader is shown in figs. 18 and 25, which were secured by placing two parallel slits symmetrically in the focal plane, that is, one on either side of the centre of the field.

We proceed to consider the explanation of the black lines marking the positions of the boundaries in the four photographs mentioned in the preceding para. In the focal plane we have two apertures extending from ξ_1 to ξ_2 and from $-\xi_1$ to $-\xi_2$ respectively. On account of the symmetry, the Ci-functions disappear from the expression for the intensity of the field as viewed in the direction ϕ, which may be written in the form

$$ I = 4\left[\text{Si} \left\{ \frac{2\pi\theta}{\lambda}\left(1+\frac{\phi}{\theta}\right)\xi_2 \right\} - \text{Si} \left\{ \frac{2\pi\theta}{\lambda}\left(1+\frac{\phi}{\theta}\right)\xi_1 \right\} \right.$$
$$ \left. + \text{Si} \left\{ \frac{2\pi\theta}{\lambda}\left(1-\frac{\phi}{\theta}\right)\xi_2 \right\} - \text{Si} \left\{ \frac{2\pi\theta}{\lambda}\left(1-\frac{\phi}{\theta}\right)\xi_1 \right\} \right], $$

where θ is the angular semi-aperture of the lens. When $\phi/\theta = \pm 1$, that is, at the boundaries, this expression becomes very small but the suddenness with which the illumination falls to zero at

these points depends very much on the magnitudes of ξ_1 and ξ_2. To illustrate this statement, I have calculated the distribution of intensity for a hypothetical case in which $\xi_1 = \dfrac{\lambda}{\theta}$ and $_2 = \dfrac{50\lambda}{\theta}$. The values are shown in Table IV (in which the factor 4 in the expression for the intensity has been neglected) and these have been plotted in fig. 3.

TABLE IV.

ϕ/θ	1	ϕ/θ	1
1·000	0·0045	1·000	0·0045
1·002	1·7956	0·998	2·2201
1·005	1·5129	0·995	1·8496
1·010	2·9584	0·990	3·4225
1·020	1·5376	0·980	2·1609
1·030	2·0164	0·970	2·4025
1·100	0·8281	0·900	1·0050
1·300	0·0020	0·700	0·0000
1·500	0·0605	0·500	0·0357
1·800	0·0060	0·200	0·0057
2·000	0·0182	0·000	0·0852

Another case in which the disparity between ξ_1 and ξ_2 was much smaller was chosen for experimental verification of the positions of the diffraction maxima and minima given by theory. It is found that the illumination falls off to practically zero value on the boundaries but much less suddenly than in the case previously discussed. The experimental data, the calculated intensities of the illumination and the theoretical and experimentally observed positions of the maxima and minima are shown in Table V. The agreement is fairly satisfactory. For comparison with the preceding case, the illumination curve has been plotted in fig. 4. As in the case of the circular boundary, the minima of illumination are absolute zeroes.

4. *Other forms of Boundary.*

The cases in which the surface is bounded by forms of apertures other than those considered previously are of interest from the point of view of the general theory of diffraction. Figs. 19 and 22 in the
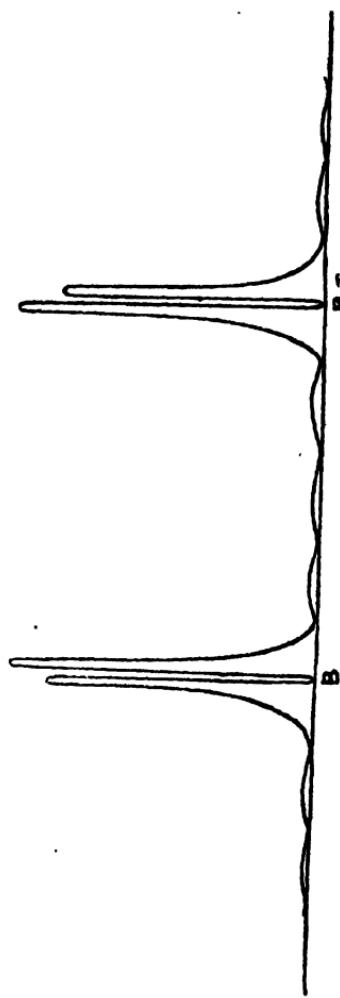
Fig. 3.

TABLE V.

$\xi_1 = \cdot654$mm., $\xi_2 = \cdot963$mm., $f = 13\cdot4$cms., $\lambda = \cdot00045$mm., width of the aperture $= \cdot0955$mm.

| ϕ/θ	$\sqrt{I}$	I (× const. factor).	Calculated values of ϕ/θ for max. & min.	Observed values of ϕ/θ for max. & min.	ϕ/θ	$\sqrt{I}$	I (× const. factor).	Calculated values of ϕ/θ for max. & min.	Observed values of ϕ/θ for max.;& min.
·760	−·0002	0	·760 min.	·756	1·000	−·0145	2	1·000 min.	1·000
·780	+·1468	204			1·005	−·0918	84		
·800	+·2123	449	·800 max.	·802	1·010	−·1528	233		
·820	+·1463	201			1·020	−·2587	669		
·840	−·0018	0	·840 min.	·843	1·030	−·3701	1370		
·860	−·1601	256			1·040	−·6898	4758	1·04 max.	1·038
·880	−·3084	957	·888 max.	·882	1·050	−·3648	1831		
·900	−·2369	559			1·060	−·2517	634		
·920	−·0065	0	·923 min.	·927	1·070	−·1513	229		
·930	+·1337	179			1·080	−·0059	0	1·080 min.	1·076
·940	+·2386	567			1·100	+·2431	589		
·950	+·3416	1167			1·120	+·3196	1021	1·114 max.	1·115
·960	+·6707	4498	·960 max.	·961	1·140	+·1713	293		
·970	+·3431	1177			1·160	−·0024	0	1·16 min.	1·157
·980	+·2398	575			1·180	−·1642	270		
·990	+·1350	182			1·200	−·2128	453	1·198 max.	1·205
·995	+·0620	38			1·220	−·1478	218		
1·000	−·0145	2	1·000 min.	1·000	1·240	−·0002	0	1·24 min.	1·233

Fig. 4.

Plate represent photographs of the effect observed when the surface is bounded by a quadrilateral and triangular apertures respectively. These photographs were obtained by using a point source of light and an annular aperture placed symmetrically in the focal plane. It will be noticed that the boundaries appear as black lines with luminous fringes on either side, thus showing a complete analogy with the cases of the circular and rectangular boundaries previously considered.

In order more fully to study the luminosity at the boundaries of the triangular, quadrilateral and other forms of aperture, an arrangement was devised in which a screen containing a small circular hole could be placed excentrically in the focal plane and rotated in this plane. This hole comes successively over different parts of the diffraction pattern formed at the focus, and the luminosity at the boundaries observed through it undergoes a series of changes. For instance, with a triangular aperture, it is known that the diffraction pattern at the focus consists of a six-rayed 'star', the 'rays' being perpendicular to the three sides of the triangle respectively. When the hole comes over any one of the rays, the corresponding boundary appears luminous, but in other cases it becomes practically invisible. Similarly with a quadrilateral aperture, the diffraction pattern is a 'star' with eight rays perpendicular to its four sides, and each of the sides appears luminous when the excentrically placed hole in the focal plane comes over the corresponding ray of the pattern.

Whether any particular part of the boundary appears luminous or not seems in general to depend on the normal to the boundary at that point being parallel to the radius vector from the centre of the focal plane to the aperture in the screen through which it is viewed. This is stated here as an experimental fact, the detailed mathematical explanation of which is deferred till a future occasion. An interesting illustration of its generality is furnished by the observation, that minute irregularities in the boundary often appear luminous when the adjoining parts which are straight are invisible from any given point in the focal plane. A discussion of the cases in which the boundary is a complicated figure such as a grating or a series of parallel apertures is also reserved for a future occasion.

5. *On the Theory of Foucault's Test with non-linear Retardations.*

Lord Rayleigh has considered in his paper the cases in which the wave before reaching the focus suffers retardations of a linear type over its various parts. The principal result obtained by him is that any direction in which the retardation has a discontinuous change is strongly marked by excess of brightness. If, however, the retardations are of *a non-linear* type, the theory becomes considerably more complicated. The various parts of the wave would arrive at foci not all lying in the plane of the knife-edge. As a first step towards a detailed discussion of such cases, I propose in this section to consider the effects noticed when the retardations are non-linear but of such magnitudes that the geometrical foci of the different parts of the wave are coincident but lie in a plane in advance of or behind the plane of the knife-edge. Practically speaking, this case is identical with that in which the wave suffers no retardations whatever but the knife-edge is introduced in a plane lying behind or in advance of the focus. Actual experimental observations in the case of a rectilinear boundary under these conditions show that when the knife-edge is sufficiently advanced so as to cut off the geometrical pencil of rays, the two edges of the field still appear luminous but *differ markedly in their brilliancy.* This difference becomes greater and greater as the knife-edge is put in a plane farther and farther from the focus. The effect is reversed if the knife-edge is put in front of instead of behind the focal plane and has no doubt been frequently noticed by those who have applied Foucault's test in practice. It should be remarked also that when the knife-edge is considerably advanced in any given plane, the inequality of the luminosity of the two edges diminishes. The mathematical theory of these effects is given below.

Let A represent the lens with its rectangular aperture which brings parallel rays to a focus O. Let a screen containing an aperture parallel to the aperture of the lens be placed at a distance a in advance of the focus. A is the centre of the first aperture and Q any point on it, so that $AQ = s$. If P be any point on the second aperture and B the centre of the screen, then putting $BP = \xi$ and $AO = f$, we get

$$PQ^2 = QO^2 + OP^2 - 2QO.OP \cos POQ$$

$$= f^2 + a^2 + \xi^2 + 2f . \sqrt{a^2 + \xi^2} \cos \left(\frac{s}{f} + \tan^{-1} \frac{\xi}{a} \right)$$

$$= (f + \sqrt{a^2 + \xi^2})^2 - 4f . \sqrt{a^2 + \xi^2} \sin^2 \tfrac{1}{2} \left(\frac{s}{f} + \tan^{-1} \frac{\xi}{a} \right).$$

Since $\sin^2 \tfrac{1}{2} \left(\frac{s}{f} + \tan^{-1} \frac{\xi}{a} \right)$ is small provided $\frac{\xi}{a}$ is small, we have, extracting the square root

$$PQ = f + \sqrt{a^2 + \xi^2} - \frac{2f . \sqrt{a^2 + \xi^2}}{f + \sqrt{a^2 + \xi^2}} \sin^2 \tfrac{1}{2} \left(\frac{s}{f} + \tan^{-1} \frac{\xi}{a} \right).$$

Now, assuming that ξ/a is small, we get

$$PQ = f + \sqrt{a^2 + \xi^2} - \frac{f \sqrt{a^2 + \xi^2} \cdot \xi^2}{2 . f + \sqrt{a^2 + \xi^2} , a^2} - \frac{\xi \sqrt{a^2 + \xi^2}}{a(f + \sqrt{a^2 + \xi^2})} .$$

$$- \frac{\sqrt{a^2 + \xi^2}}{2f (f + \sqrt{a^2 + \xi^2})} . s^2 .$$

Thus we see that

$$PQ = f + A + Bs + Cs^2,$$

where

$$A = \sqrt{a^2 + \xi^2} - \frac{\sqrt{a^2 + \xi^2} \cdot f \cdot \xi^2}{2 (f + \sqrt{a^2 + \xi^2}) a^2},$$

$$B = - \frac{\xi \sqrt{a^2 + \xi^2}}{a (f + \sqrt{a^2 + \xi^2})} ,$$

$$C = - \frac{\sqrt{a^2 + \xi^2}}{2f . (f + \sqrt{a^2 + \xi^2})} .$$

Therefore the vibration at the point ξ of the second aperture will be represented by

$$\int_{-s}^{s} \cos 2\pi \left(\frac{t}{\tau} - \frac{f + A + Bs + Cs^2}{\lambda} \right) ds.$$

If T be written for $\left(\frac{t}{\tau} - \frac{f}{\lambda} \right) \lambda$, the above integral can be written in the form

$$\cos \frac{2\pi}{\lambda} T \int_{-s}^{s} \cos 2\pi \left(\frac{A + Bs + Cs^2}{\lambda} \right) ds$$

$$+ \sin \frac{2\pi}{\lambda} T \int_{-s}^{s} \sin 2\pi \left(\frac{A + Bs + Cs^2}{\lambda} \right) ds,$$

6

The rays from the various points of the second aperture may be regarded as a parallel pencil inclined to the axis at a small angle ϕ. When we proceed to enquire what is to be observed at an angle ϕ, we have to consider the expression

$$\text{Cos } \kappa \text{ T} \int_{\xi_1}^{\xi_2} \int_{-s}^{s} \cos \kappa \ (A+Bs+Cs^2-\phi\xi) \ ds \ d\xi$$

$$+\sin \kappa \text{ T} \int_{\xi_1}^{\xi_2} \int_{-s}^{s} \sin \kappa \ (A+Bs+Cs^2-\phi\xi) \ ds d\xi,$$

where $\kappa = \dfrac{2\pi}{\lambda}$ and ξ_1, ξ_2 define the limits of the second aperture.

The intensity I represented as the sum of the squares of the integrals, is given by

$$I=\left[\int_{\xi_1}^{\xi_2} \int_{-s}^{s} \cos \kappa \ (A+Bs+Cs^2-\phi\xi) \ ds d\xi \ \right]^2$$

$$+\left[\int_{\xi_1}^{\xi_2} \int_{-s}^{s} \sin \kappa \ (A+Bs+Cs^2-\phi\xi) \ ds d\xi \ \right]^2.$$

The integration can be carried out with respect to s on putting $s=z-a$ and choosing a, so that the term containing the first power of z in the expression $A+B(z-a)+C(z-a)^2-\phi\xi$ vanishes. The integrals are thus reduced to integrals of the Fresnel class and can be integrated in semi-convergent series. Since κ is a large quantity, we retain only a few terms of the series, and the subsequent integration with respect to ξ is effected by integrating by parts.

We thus find that the intensity is proportional to the sum of the squares of the expressions (I) and (II) given below

(1) $\dfrac{c_2}{2C_2\kappa}\left[-\cos \kappa \ (A_2+B_2 s+C_2 s^2-\phi\xi_2) \ s \right]$

$$+\left\{ a_2-(b_2-\frac{B_2 c_2}{2C_2})\frac{B_2}{2C_2} \right\}\left[\cdot \ \frac{\cos \kappa \ (A_2+B_2 s+C_2 s^2-\phi\xi_2)}{\kappa \ (B_2+2C_2 s)} \right]$$

$$+(b_2-\frac{B_2 c_2}{2C_2^2})\frac{1}{C_2\kappa}\left[-\cos \kappa \ (A_2+B_2 s+C_2 s^2-\phi\xi_2) \right]_{-s}$$

$$-\frac{c_1}{2C_1\kappa}\left[-\cos\kappa\,(A_1+B_1s+C_1s^2-\phi\xi_1)\cdot s\right]_{-s}^{s}$$

$$-\left\{a_1-\left(b_1-\frac{B_1c_1}{2C_1}\right)\frac{B_1}{2C_1}\right\}\left[-\frac{\cos\kappa(A_1+B_1s+C_1s^2-\phi\xi_1)}{\kappa\,(B_1+2C_1s)}\right]_{-s}^{s}$$

$$-\left(b_1-\frac{B_1c_1}{2C_1}\right)\frac{1}{2\,C_1\kappa}\left[-\cos\kappa\,(A_1+B_1s+C_1s^2-\phi\xi_1)\right]_{-s}^{s}$$

$$(11)-\frac{c_2}{2\,C_2\kappa}\left[\sin\kappa\,(A_2+B_2s+C_2s^2-\phi\xi_2)\cdot s\right]_{-s}^{s}$$

$$-\left\{a_2-\left(b_2-\frac{B_2c_2}{2C_2}\right)\frac{B_2}{2\,C_2}\right\}\left[\frac{\sin\kappa\,(A_2+B_2s+C_2s^2-\phi\xi_2)}{\kappa\,(B_2+2\,C_2s)}\right]$$

$$-\left(b_2-\frac{B_2c_2}{2C_2}\right)\frac{1}{2\,C_2\kappa}\left[\sin\kappa\,(A_2+B_2s+C_2s^2-\phi\xi_2)\right]_{-s}^{s}$$

$$+\frac{c_1}{2\,C_1\kappa}\left[\sin\kappa\,(A_1+B_1s+C_1s^2-\phi\xi_1)\cdot s\right]_{-s}^{s}$$

$$\left.a_1-\left(b_1-\frac{B_1c_1}{2C_1}\right)\frac{B_1}{2\,C_1}\right\}\left[\frac{\sin\kappa\,(A_1+B_1s+C_1s^2-\phi\xi_1)}{\kappa\,(B_1+2\,C_1s)}\right]$$

$$\left(b_1-\frac{B_1c_1}{2C_1}\right)\frac{1}{C_1\kappa}\left[\sin\kappa\,(A_1+B_1s+C_1s^2-\phi\xi_1)\right]_{-s}^{s},$$

where

$$A_2=a+\tfrac{1}{2}\frac{1}{f+a}\,\xi_2^{2},$$

$$B_2=-\frac{1}{f+a}\,\xi_2,$$

$$C_2=-\frac{1}{2f\,(f+a)}\left[1+\frac{f}{2a^2\,(f+a)}\,\xi_2^{2}\right],$$

$$A_1=a+\tfrac{1}{2}\frac{1}{f+a}\,\xi_1^{2},$$

$$B_1=-\frac{1}{f+a}\xi_1,$$

$$C_1 = -\frac{a}{2f(f+a)}\left[1+\frac{f}{2u^3(f+a)}\xi_1\right],$$

$$a_2 = \frac{f+a}{\xi_2 - \phi(f+a)},$$

$$b_2 = \frac{f+a}{\{\xi_2 - \phi(f+a)\}^3},$$

$$\frac{f+a}{\{\xi_2 - \phi(f+a)\}^3} + \frac{\xi_2(f+a)}{2a}\frac{1}{f+a\cdot\{\xi_2 - \phi(f+a)\}}$$

$$a_1 = \frac{f+a}{\xi_1 - \phi(f+a)},$$

$$b_1 = \frac{f+a}{\{\xi_1 - \phi(f+a)\}^3},$$

$$c_1 - \frac{f+a}{\{\xi_1 - \phi\, f+a)\}^3} + \frac{\xi_1(f+a)}{2a(f+a)\{\xi_1 - \phi(f+a)\}^3}.$$

This expression has been used to determine the ratio of the intensities of the two edges for different positions of the advancing edge at different distances from the focal plane. The ratio of the brightness of the two edges was also determined experimentally by photometric comparison. For this purpose, a double-image prism was used to obtain a pair of images of the luminous edges polarised in perpendicular planes which were then observed through a nicol. The results are shown in table VI. The agreement between theory and experiment is fairly satisfactory.

TABLE VI.

$2s = 3\cdot47$ mm., $f = 43\cdot3$ cm., $\lambda = \cdot0006$ mm.

a	ξ_1	ξ_2	Observed ratio of the intensity of the two edges.	Calculated ratio of the intensity of the two edges.
−4·0 cm.	0·85 mm.	5·65 mm.	0·25	0·21
−4·0 ,,	2·65 ,,	5·65 ,,	0·41	0·38
−2·5 ,,	0·72 ,,	5·38 ,,	0·32	0·30
−2·5 ,,	2·43 ,,	5·38 ,,	0·53	0·48
−1·5 ,,	1·00 ,,	5·95 ,,	0·50	0 45
−1·5 ,,	2·87 ,,	5·82 ,,	0·65	0·59
0 ,,	...	...	1·00	1·00
+2·5 ,,	·76 ,,	5·76 ,,	1·32	1·28
+2·5 ,,	2·76 ,,	5·76 ,,	1·14	1·11
+4·5 ,,	·61 ,,	5·61 ,,	2·37	2·29
+4·5 ,,	2·6 ,,	5·61 ,,	1·42	1·37

6. On the Emission of Light by the Edges of Diffracting Apertures.

A few further observations on the apparent emission of light from the edges of a diffracting aperture may be mentioned here. In the previous sections, it has been shown that the boundary of a diffracting surface appears *as a black line* (surrounded by luminous fringes on either side), when observed solely by diffracted light admitted through apertures placed *symmetrically* in the focal plane. A similar result is also found when the screen which blocks out the geometrical pencil of rays is placed symmetrically in any plane either in or beyond the focus. It would appear from this observation that we may regard the edge of a diffracting aperture as sending out two streams of light in directions more or less nearly normal to itself (one on each side of the wave normal), and that these streams are in *opposite phases.* This is distinctly suggested by Sommerfeld's well-known investigation on the diffraction of light by semi-infinite screen, though of course the results are not strictly applicable in the case of a convergent wave limited by rectilinear boundaries at a finite distance from each other. In this investigation[1] it is shown that the illumination in any part of the field (excluding a very limited region on either side of the dividing planes) may be found by superposing upon the direct and reflected streams of light, a radiation emitted by the edge of the screen, the amplitude of this being given by the expression

$$\frac{1}{4\pi}\cos\left\{2\pi\left(\frac{r}{\lambda}-\frac{l}{\tau}\right)+\frac{\pi}{4}\right\}\sqrt{\frac{\lambda}{r}}\left[\frac{+1}{\cos\frac{\phi+\phi'}{2}}-\frac{1}{\cos\frac{\phi-\phi'}{2}}\right].$$

When ϕ is nearly equal to $\pi-\phi'$ or $\pi+\phi'$, one of the two terms inside the square brackets becomes very large compared with the other, and the latter may then be neglected. When ϕ changes from a value less than $\pi-\phi'$ to a value greater than $\pi-\phi'$ or if it changes from a value less than $\pi+\phi'$ to a value greater than $\pi+\phi'$, the expression for the amplitude changes sign, showing that the phase of the radiation emitted by the edge changes by π when we pass from one side of the plane of transition to the other.[2]

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[1] Math. Ann., Bd. 47, (1896).

[2] For the case of the rectangular boundary the change in the phase of the vibration *at the focal plane* in passing across the zero point has been remarked upon by Lord Rayleigh in his paper. Phil. Mag., Feb., 1917, p. 171.

I have also attempted to study the phenomena of the flow of energy in the optical field due to rectangular and circular boundaries in convergent-light by placing a narrow screen, or a wide screen with a narrow aperture in it in some selected part of the field and tracing the phenomena observed in its rear. Interesting results have been noticed particularly in the case of a circular boundary but a description of this is reserved for a future occasion.

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Calcutta, the 24th January, 1918.

ON THE FORCED VIBRATIONS OF A HETEROGENEOUS STRING

BY

SUDHANSUKUMAR BANERJI.

1. The boundary problems in one dimension had already a considerable literature* by the end of the nineteenth century, which since that time had expanded rapidly. By such a problem is understood primarily the following question† :—to determine whether an ordinary differential equation has one or more solutions which satisfy certain boundary conditions, and if so, what the character of these solutions

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* In connection with this problem I have consulted the following books. memoirs and papers :—

1. *Liouville's Journal*, Vol. I, (1836), pp. 106-186; Vol. II, (1837), pp. 16 and 418; Vol. III, (1838) p. 604; Vol. V, (1840), p. 356 ; Vol. III, (1838), p. 361.

2. Kneser, *Die Integral Gleichungen*, pp. 52-53 and the chapter on the theorems of Sturm and Liouville. Also, *Math. Ann.*, Vol. 58, p. 81 ; Vol. 63, p. 477 : Vol. 61, p. 402; Vol. 42, p. 409.

3. Forsyth & *Theory of Differential Equations*, Part III, Vol. IV, Chapter IX.

4. Rayleigh's *Theory of Sound*, Chaps. V and VI.

5. Bocher's *Introduction to the study of Integral Equations*. Also *Bull. Amer. Math. Soc.*, Vol. 18 (1911), p. I ; Vol. 7, (1901), p. 297 and p. 333 ; *Trans. Amer. Math. Soc.*, Vol. I, (1900), p. 414.

6. G. W. Hill, *Acta Mathematica*, Vol. VIII, (1886).

7. Liapounoff, *Comptes Rendus*, t. CXXIII (1896), pp. 1248-1252, t. CXXVIII (1898), pp. 910-913, 1085-1090.

8. Mason, *Math. Ann.*, Vol. 58, (1904), p. 532 and p. 528; *Trans. Amer. Math. Soc.*, Vol. 7, (1906), p. 340 also Vol. 13, p. 516, and Vol. 8, (1907), p. 431.

9. Hilbert, *Göttinger Nachrichten*, 1904.

10. Schmidt, *Math. Ann.*, Vol. 63, (1907), p. 433.

11. Birkhoff, *Trans. Amer. Math. Soc.*, Vol. 10, p. 259; Vol. 9, p. 373.

12. Dunkel, *Bull. Amer. Math. Soc.*, Vol. 8, (1902), p. 288.

13. Hobson, *Proc. Lond. Math. Soc.*, 2nd series, Vol. 6, p. 349.

14. Horn, *Math. Ann.*, Vol. 52, p. 271 and p. 340.

15. Holmgren, *Arkiv. för Mat. Astr. Och Fysik*, Vol. I, (1904), p. 401.

16. Richardson, *Math. Ann.*, Vol. 68, (1910), p. 279.

17. A. C. Dixon, *Proc. Lond. Math. Soc.*, Series 2, Vol. 3, p. 83.

18. Haar, Zur Theorie der orthogonalen Funktionen systeme, *Math. Ann.*, Vol. 69, (1910), p. 331 ; Vol. 71, (1911), p. 38.

† See Bocher's Lecture on Boundary Problems in one Dimension, *Fifth International Congress of Mathematicians*, (1912), Vol. II, p. 163.

is and how their character changes when the differential equation or the boundary conditions change. This problem is one of which special cases come up constantly in applied mathematics. In the present paper, I propose to consider in detail from the point of view of physical applications one such case, namely, the problem of the forced vibrations of a heterogeneous string. I have also considered a few illustrative examples in which the laws of density have been so chosen that the solutions assume extremely simple forms.

It is well-known that the differential equation of free vibrations,

$$\frac{d^2 y}{dx^2} + \lambda^2 \, \rho\,(x) \, y = 0, \tag{1}$$

together with a system of boundary conditions can be replaced by a single integral equation of the second kind which is of the form

$$v(x) = f(x) + \lambda^2 \int_a^b G(x, \xi) u(\xi) d\xi.$$

Kneser in his book on integral equations has given a solution of the present problem for the case when the differential equation (1) is equivalent to *an integral equation of the first kind* with *a symmetrical kernel, viz.*, to an integral equation of the form

$$u(x) = \lambda^2 \int_a^b K(x, \xi) u(\xi) d\xi,$$

where $K(x, \xi)$ is symmetrical in x, ξ.

The solution given in the present paper is slightly different in form from that given by Kneser and is more general.

2. The differential equation of the forced vibrations of a string whose logitudinal density $\rho(x)$ is variable and which is under the action of a transverse force χ is

$$\rho(x) \frac{d^2 y}{dt^2} = T \frac{d^2 y}{dx^2} + \chi,$$

or simply

$$\rho(x) \frac{d^2 y}{dt^2} = \frac{d^2 y}{dx^2} + \chi. \tag{2}$$

if T be supposed to be included in $\rho(x)$ and χ.

If the string be supposed to be fixed at its extremities, we must have

$$y = 0, \text{ when } x = 0,$$

and $y = 0, \text{ when } x = l,$ \hfill (3)

where l denotes the length of the string.

Now taking the differential equation of free vibrations of the string, *viz.*,

$$\frac{d^2y}{dx^2}+\lambda_n{}^2\rho(x)y=0,\qquad(1)$$

where as is well-known λ_1, λ_2, etc., determine the different free periods of the string, we see that if u_r, u_s be two different normal functions satisfying the equation, then

$$\frac{d^2u_r}{dx^2}+\lambda_r{}^2\rho(x)u_r=0,$$

$$\frac{d^2u_s}{dx^2}+\lambda_s{}^2\rho(x)u_s=0.$$

Multiplying the second by u_r and the first by u_s and intergrating with respect to x between the limits o to $\cdot$, we get

$$u_s\frac{du_r}{dc}-u_r\frac{du_s}{dc}=(\lambda_s{}^2-\lambda_r{}^2)\int_o^s\rho(x)u_r u_s dx.$$

But since both u_r and u_s vanish when $=o$ and $r=l$, we get

$$\int_o^l\rho(x)u_r u_s dc=o.\qquad(4)$$

Now it is possible to express the solution of (2), *viz.*,

$$\rho(\cdot)\frac{d^2y}{dt^2}=\frac{d^2y}{dc^2}+\chi,\qquad(2)$$

in a series of these normal functions, that is to say, we can take

$$y=\sum q_n u_n(c).\qquad(5)$$

Substituting in the differential equation, we get

$$\rho(x)\sum_n u_n(x)\frac{d^2q_n}{dt^2}=\sum_n q_n\frac{d^2u_n(\cdot)}{dx^2}+\chi.$$

(This will be true only if it is possible to differentiate (5) term by term. In cases where this not possible this process will not be valid. For these cases a different method of procedure will have to be adopted.)

Now since

$$\frac{d^2u_n(x)}{dx^2}+\lambda_n{}^2\rho(\cdot)u_n(c)=o,$$

we get

$$\rho(x)\sum_n u_n(x)\frac{d^2q_n}{dt^2}=-\sum_n q_n\lambda_n{}^2\rho(x)u_n(x)+\chi$$

or,

$$\sum_n u_n(v)\left\{\frac{d^2q_n}{dt^2}+\lambda_n{}^2q_n\right\}=\frac{\chi}{\rho(\varpi)}.\qquad(6)$$

Expanding $\dfrac{\chi}{\rho(x)}$ in a series of the same normal functions, we get

$$\frac{\chi}{\rho(x)}=a_0u_0(x)+a_1u_1(x)+\ldots+a_nu_n(x)+\ldots,$$

where $a_n=\dfrac{\displaystyle\int_o^l\chi\,u_n(x)dx}{\displaystyle\int_o^l\rho(x)u_n^2(x)dx}$.

(Since $\rho(v)$ is finite and continuous between the limits o to l and can never be zero between these limits, it will in general be possible to expand $\dfrac{\chi}{\rho(x)}$ in the above form.)

In other words

$$\frac{\chi}{\rho(x)}=\sum_n u_n(x)\,\frac{\displaystyle\int_o^l\chi u_n(x)dx}{\displaystyle\int_o^l\rho(x)u_n^2(x)dx}.$$

Substituting this expression in (6), we get

$$\sum_n u_n(x)\left\{\frac{d^2q_n}{dt^2}+\lambda_n^2q_n\right\}=\sum_n u_n(x)\,\frac{\displaystyle\int_o^l\chi u_n(x)dx}{\displaystyle\int_o^l\rho(x)u_n^2(x)dx}.$$

Equating the co-efficients of $u_n(x)$, we get

$$\frac{d^2q_n}{dt^2}+\lambda_n^2q_n=\frac{\displaystyle\int_o^l\chi u_n(x)dx}{\displaystyle\int_o^l\rho(x)u_n^2(x)dx}.$$

Now writing $Q_n=\dfrac{\displaystyle\int_o^l\chi u_n(x)dx}{\displaystyle\int_o^l\rho(x)u_n^2(x)dx}$, we have $\dfrac{d^2q_n}{dt^2}+\lambda_n^2q_n=Q_n.$ (7)

If we suppose that χ is a simple harmonic periodic function of the time, so that

$$\chi=F(x)\cos(\beta t+\gamma),$$

we must have also

$$y=u\cos(\beta t+\gamma).$$

We have '

$$Q_n=\odot_n\cos(\beta t+\gamma),$$

where

$$\Theta_n = \frac{\int_0^l F(x)u_n(x)dx}{\int_0^l \rho(x)u_n^2(x)dx} .$$

Now putting $q_n = \theta_n \cos(\beta t + \gamma)$,

we get from (7), $(\lambda_n^2 - \beta^2)\theta_n = \Theta_n$.

Thus we get

$$y = \Sigma q_n u_n(x) = \Sigma \theta_n u_n(x)\cos(\beta t + \gamma) = u \cos(\beta t + \gamma),$$
$$\quad\quad n \quad\quad\quad\quad n$$

$$u = \Sigma \theta_n u_n(x) = \Sigma \frac{\Theta_n u_n(x)}{\lambda_n^2 - \beta^2} = \Sigma \frac{u_n(x)}{\lambda_n^2 - \beta^2} \frac{\int_0^l F(x)u_n(x)dx}{\int_0^l \rho(x)u_n^2(x)dx} .$$
$$\quad\quad n \quad\quad\quad\quad u \quad\quad\quad\quad\quad n$$

Therefore

$$y = \cos(\beta t + \gamma) \Sigma \frac{u_n(x)}{\lambda_n^2 - \beta^2} \frac{\int_0^l F(x)u_n(x)dx}{\int_0^l \rho(x)u_n^2(x)dx} . \qquad (8)$$
$$\quad\quad\quad\quad\quad n$$

This expression represents the solution of the differential equation of forced vibrations, if $u_n(x)$ is a normal function satisfying the differential equation of free vibrations, viz.,

$$\frac{d^2y}{dx^2} + \lambda_n^2\rho(x)y = 0.$$

In particular cases, the solution may assume very simple forms.

For example, if in the expansion of $\frac{F(x)}{\rho(x)}$, viz., in

$$\frac{F(x)}{\rho(x)} = a_0 u_0(x) + a_1 u_1(x) + \ldots + a_n u_n(x) + \ldots,$$

the terms are found to be rapidly diminishing in value so that we can retain a few terms (say three) and neglect the rest, the solution can be simply expressed as

$$y = \cos(\beta t + \gamma) \left[a_0 \frac{u_0(x)Q_0}{\lambda_0^2 - \beta^2} + a_1 \frac{u_1(x)Q_1}{\lambda_1^2 - \beta^2} + a_2 \frac{u_2(x)Q_2}{\lambda_2^2 - \beta^2} \right].$$

The solution of the differential equation of the forced vibrations of the string given by expression (8) gives an explicit relation between the normal functions for the string, the density and the forces acting on the string, and is extremely useful for the purposes physical applications.

3. We shall next obtain the integral equation satisfied by a solution of the differential equation

$$\rho(x)\frac{d^2y}{dt^2} = \frac{d^2y}{dv^2} + \chi,$$

and express the solution in a slightly different but interesting form.

If we write the density function in the form

$$\rho(r) = 1 - \sigma(r),$$

then the differential equation of free vibrations can be expressed in the form

$$\frac{d^2y}{dx^2} + \lambda^2[1-\sigma(x)] y = 0. \tag{9}$$

If ϕ $(\cdot)$ is a continuous function, then it is well-known that the general solution of

$$\frac{d^2y}{dt^2} + \lambda^2 y = \phi(x),$$

is

$$y\ (r) = a \sin \lambda\ (x-a) + \beta \cos \lambda\ (\cdot -a) + \frac{1}{\lambda}\int_a^x \phi(\xi) \sin \lambda(x-\xi)\ d\xi. \tag{10}$$

Now if we denote by u (r) a solution of the equation

$$\frac{d^2y}{dv^2} + \lambda^2\ [1-\sigma\ (x)] y = 0,$$

then u (r) will also be solution of the non-homogeneous equation

$$\frac{d^2y}{dv^2} + \lambda^2 y = \lambda^2\sigma(r)\ u\ (x).$$

Consequently, by using (10) we get

$$u\ (x) = a \sin \lambda\ (x-a) + \beta \cos \lambda\ (x-a)$$

$$+ \lambda \int_a^x \sigma\ (\xi) \sin \lambda\ (x-\xi)\ u\ (\xi)\ d\xi.$$

Thus we see that the differential equation of free vibrations

$$\frac{d^2y}{dx^2} + \lambda^2[1-\sigma(x)] y = 0,$$

is equivalent to the integral equation

$$u(x) = a \sin \lambda (x-a) + \beta \cos \lambda (x-a)$$

$$+ \lambda \int_a \sigma(\xi) \sin \lambda (x - \xi) u(\xi) d\xi.$$

But the boundary conditions give $u(x) = o$, when $x = o$ and $u(x) = o$ when $x = l$.

So that we must have

$$u(x) = a \sin \lambda x + \lambda \int_0^x \sigma(\xi) \sin \lambda (x - \xi) u(\xi) d\xi.$$

(The condition $u(x) = o$, when $x = l$, is satisfied for the λ's which determine the free periods are the roots of the equation $u(l) = o$.)

Thus we see that the normal function $u_n(x)$ satisfies the equation

$$u_n(x) = a_n \sin \lambda_n x + \lambda_n \int_0 \sigma(\xi) \sin \lambda_n (x - \xi) u_n(\xi) d\xi. \quad (11)$$

We have seen that the solution of the differential equation of forced vibrations can be expressed in the form

$$y = \cos(\beta t + \gamma) \sum_n \frac{u_n(x)}{\lambda_n^2 - \beta^2} \cdot Q_n,$$

where

$$Q_n = \frac{\int_o^l F(x) u_n(x) dx}{\int_o^l \rho(x) u_n^2(x) dx}.$$

Putting

$$\psi(x) = \sum_n \frac{u_n(x)}{\lambda_n^2 - \beta^2} \cdot Q_n, \text{ we get on multiplying}$$

by $\sigma(\xi) \dfrac{\sin \lambda_n(x - \xi)}{\lambda_n}$ and integrating

$$\int_0^x \sigma(\xi) \frac{\sin \lambda_n(x-\xi)}{\lambda_n} \psi(\xi) d\xi = \sum_n \frac{Q_n}{\lambda_n^2 - \beta^2} \int_0^x \sigma(\xi) \frac{\sin \lambda_n(x-\xi)}{\lambda_n} u_n(\xi) d\xi.$$

But since

$$u_n(x) = a_n \sin \lambda_n x + \lambda_n^2 \int_0^x \sigma(\xi) \frac{\sin \lambda_n(x-\xi)}{\lambda_n} u_n(\xi) d\xi,$$

we get

$$\int_t^t \sigma(\xi)\frac{\sin\lambda_n(x-\xi)}{\lambda_n}\ \psi(\xi)d\xi$$

$$=\sum_n \frac{u_n(x)-a_n\sin\lambda_n x}{(\lambda_n^2-\beta^2)\lambda_n^2}\cdot Q_n$$

$$=\sum_n \frac{1}{\beta^2}\left[u_n(x)-a_n\sin\lambda_n x\right]\left(\frac{1}{\lambda_n^2-\beta^2}-\frac{1}{\lambda_n^2}\right)Q_n$$

$$=\frac{\psi(x)}{\beta^2}-\frac{1}{\beta^2}\sum_n \frac{u_n(x)}{\lambda_n^2}Q_n-\sum_n a_n\frac{\sin\lambda_n x}{(\lambda_n^2-\beta^2)\lambda_n^2}Q_n.$$

If we denote by $f(x)$ the expression

$$f(x)=\sum_n \frac{u_n(x)}{\lambda_n^2}\cdot Q_n-\beta^2\sum_n a_n\frac{\sin\lambda_n x}{\lambda_n^2(\lambda_n^2-\beta^2)}\cdot Q_n,$$

we find that $\psi(x)$ satisfies the integral equation

$$\psi(x)=f(x)+\lambda\int_0^v \sigma(\xi)\frac{\sin\lambda_n(x-\xi)}{\lambda_n}\psi(\xi)d\xi, \tag{12}$$

where $\lambda=\beta^2$.

Thus we get

$$y=\psi(x)\cos(\beta t+\gamma)$$

$$=\cos(\beta t+\gamma)\sum_n \frac{u_n(x)}{\lambda_n^2-\beta^2}\cdot Q_n$$

$$=\cos(\beta t+\gamma)\sum_n\left[\frac{u_n(x)}{\lambda_n^2}\cdot Q_n+\frac{\beta^2 Q_n u_n(x)}{\lambda_n^2(\lambda_n^2-\beta^2)}\right]$$

or,

$$y=\cos(\beta t+\gamma)\left[f(x)+\beta^2\sum_n \frac{c_n(u_n+a_n\sin\lambda_n x)}{\lambda_n^2-\beta^2}\right],$$

where $c_n=\dfrac{1}{\lambda_n^2}\cdot Q_n=\dfrac{1}{\lambda_n^2}\dfrac{\int_0^1 F(x)u_n(x)dx}{\int_0^1\rho(x)u_n^2(x)dx}$ (13)

It is easy to see that this expression agrees with the solution given by Kneser for the case of constant density* on putting $\sigma(x)=o$.

4. The solution of the differential equation of forced vibrations given by expressions (8) and (13) involves the assumption that the

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* Kneser, *Die Integral Gleichungen*, pp. 52-5

normal functions $u_n(x)$ for the string which are solutions of the differential equation

$$\frac{d^2y}{dx^2}+\lambda_n^2\rho(x)\,y=0,$$

or of the integral equation (11) are known. But a general integration of this differential equation is beyond our powers. Although this differential equation has not hitherto been solved in finite terms, the theory* of the solution of the linear equation of the second order has been considerably developed. The theory started with Sturm who in an extensive memoir gave many remarkable results concerning the roots and oscillatory properties of the solutions of differential equations and the existence of characteristic values. Sturm's work has been developed in various directions since his time, partly by methods more or less closely related to his own and partly by a number of essentially different methods. Of these there are four which may be briefly described as :—

(1) Liouville's method of asymptotic expressions,

(2) The method of successive approximations,

(3) The minimum principle,

(4) The integral equations.

The discovery of the asymptotic expressions for the large characteristic values and the corresponding characteristic functions, and the applications of these expressions in the theory of the development of arbitrary functions as well as the method of successive approximations are due to Liouville.

The minimum principle commonly known as the Dirichlet's principle started with Weierstrass and Weber and in recent years has been fully developed by Hilbert, Holmgren and Mason. Lastly, the method by which a linear differential equation together with a system of linear boundary conditions can be replaced by a single integral equation of the second kind as well as the theory of these integral equations have formed the subject of enquiry in the last few years of many distinguished mathematicians including Hilbert, Fredholm and Volterra.

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* Bôcher on Boundary problems in one Dimension, *Fifth International Congress of Mathematicians*, (1912).

In a classical memoir on the determination of the lunar perigee, G. W. Hill has studied the nature of the solution of the differential equation

$$\frac{d^2u}{d\xi^2} + (\theta_o + 2\theta_1 \cos 2\xi + 2\theta_2 \cos 4\xi + \quad.)\,u = o,$$

the co-efficients θ_1, θ_2, . being considerably smaller than θ_o. The theory of the solutions of linear differential equations with uniform periodic co-efficients has also been developed by various writers and especially by Liapounoff whose investigations deal with the equation

$$\frac{d^2u}{dz^2} + \mu\, u\, p\,(z) = o,$$

where μ is a parameter, and $p(z)$ is a uniform periodic function of period ω.

The following method of approximating to the solution of the differential equation

$$\frac{d^2y}{d\,z^2} + \lambda^2\, \rho(z)y = o,$$

due to Lord Kelvin[*] which is of very great importance in physical applications for simple cases of density distribution is perhaps well-known.

The differential equation can be written in the form

$$y = -\int d\epsilon \int \lambda^2 \rho(z)y\, dx$$

and it is easy to prove that if

$$y_2 = -\int_o^z d\epsilon \int_o^z \lambda^2 \rho y_1 dx, \quad y_3 = -\int_o^z dx \int_o^z \lambda^2 \rho(x)y_2 dr, \text{ etc.,}$$

where y_1 is any function of r, to begin with, as for example $y_1 = r$, then y_2, y_3, etc. are successive approximations converging to that one of the solutions of the differential equation which vanishes when $r = 0$.

We thus see that

$$y = y_o - y_1 + y_2 - \cdots,$$

where

$$y_o = Cx + C',$$

$$y_{n+1} = \int_o^x dx \int_o^x \lambda^2 \rho(x)y_n\, dx,$$

---

[*] See Proc. Roy. Soc., Vol. XXIV, (1876), p. 269.

expresses the solution of the differential equation in a series necessarily converging for all values of x, provided $\rho(x)$ remains finite, C and C' being two arbitrary constants.

When $\rho(x)$ is given, we can easily calculate successively the functions y_1, y_2, etc. For example, if we take

$$\rho(x) = \rho_0 + \rho_n x^n$$

we easily find that the solution of

$$\frac{d^2y}{dx^2} + (\rho_0 + \rho_n x^n)\, y = 0,$$

can be written in the form

$$y = y_0 - y_1 + y_2 - y_3 + \dots$$

where

$$y_0 = Cx + C'$$

$$y_1 = \int_0^x dx \int_0^x (\rho_0 + \rho_n x^n)\,(Cx + C')\, dx$$

$$= C'\rho_0 \frac{x^2}{2!} + C\rho_0 \frac{x^3}{3!} + C'\rho_n \frac{x^{n+2}}{(n+1)(n+2)} + C\rho_n \frac{x^{n+3}}{(n+2)(n+3)},$$

$$y_2 = C'\rho_0^2 \frac{x^4}{4!} + C\rho_0^2 \frac{x^5}{5!}$$

$$+ C'\rho_0\rho_n\, x^{n+4}\left[\frac{1}{1.\,2.\,(n+3)(n+4)} + \frac{1}{(n+1)(n+2)(n+3)(n+4)}\right]$$

$$+ C\rho_0\rho_n\, x^{n+5}\left[\frac{1}{1.\,2.\,3.\,(n+4)(n+5)} + \frac{1}{(n+2)(n+3)(n+4)(n+5)}\right]$$

$$+ C'\rho_n^2\, \frac{1}{(n+1)(n+2)(2n+3)(2n+4)} + C\rho_n^2\, \frac{1}{(n+2)(n+3)(2n+4)(2n+5)}$$

etc.

In the particular case, when $n = 1$, the solution of the equation

$$\frac{d^2y}{dt^2} + (\rho_0 + \rho_1 x)y = 0,$$

can be written in the form

$$y = C(\rho_0 + \rho_1 x)\left[1 - \frac{2}{4!}\frac{(\rho_0 + \rho_1 x)^3}{\rho_1^3} + \frac{2.5}{7!}\frac{(\rho_0 + \rho_1 x)^6}{\rho_1^4}\right.$$

$$\left. - \frac{2.5.8}{10!}\frac{(\rho_0 + \rho_1 x)^9}{\rho_1^6} + \dots\right]$$

$$+ C'\left[1 - \frac{1}{3!}\frac{(\rho_0 + \rho_1 x)^3}{\rho_1^2} + \frac{1.4}{6!}\frac{(\rho_0 + \rho_1 x)^6}{\rho_1^4} - \frac{1.4.7}{9!}\frac{(\rho_0 + \rho_1 x)^9}{\rho_1^6} + \dots\right].$$

5. We shall next consider a few illustrative examples in which the density function will be so .chosen that the differential equation of free vibrations can be integrated in finite terms.

Example 1.—As the simplest case, let us consider the differential equation, when

$$\rho(x) = C_0 + C_1 x.$$

It is well-known that the primitive of

$$\frac{d^2 u}{dc^2} + v^2 x^n u = 0,$$

is given by

$$u = x^{\frac{1}{2}} \left[A\, J_{\frac{1}{n+2}} \left(\frac{2v}{n+2} x^{\frac{1}{2}(n+2)} \right) + B\, J_{-\frac{1}{n+2}} \left(\frac{2v}{n+2} x^{\frac{1}{2}(n+2)} \right) \right].$$

From this we easily find that the primitive of the differential equation

$$\frac{d^2 u}{dc^2} + \lambda^2 (C_0 + C_1 r) u = 0$$

is

$$u = \left(C_0 + C_1 r \right)^{\frac{1}{2}} \left[A J_{\frac{1}{3}} \left\{ \frac{2\lambda}{3C_1} (C_0 + C_1 x)^{\frac{3}{2}} \right\} \right.$$

$$\left. + B\, J_{-\frac{1}{3}} \left\{ \frac{2\lambda}{3C_1} (C_0 + C_1 x)^{\frac{3}{2}} \right\} \right].$$

As it is, this solution is not a normal function for the string. In order that this should be a normal function for the string, it must vanish at the limits $x = 0$ and $x = l$.

Thus we see that

$$(C_0 + C_1 x)^{\frac{1}{2}} \left[A_n J_{\frac{1}{3}} \left\{ \frac{2\lambda_n}{3C} (C_0 + C_1 r)^{\frac{3}{2}} \right\} \right.$$

$$\left. + B_n J_{-\frac{1}{3}} \left\{ \frac{2\lambda_n}{3C_1} (C_0 + C_1 r)^{\frac{3}{2}} \right\} \right]$$

is a normal function for the string, provided $A_n : B_n$ is so chosen that

$$A_n J_{\frac{1}{3}} \left(\frac{2\lambda_n}{3C_1} C_0^{\frac{3}{2}} \right) + B_n J_{-\frac{1}{3}} \left\{ \frac{2\lambda_n}{3C_1} C_0^{\frac{3}{2}} \right\} = 0,$$

$$A_n J_{\frac{1}{3}} \left\{ \frac{2\lambda_n}{3C_1} (C_0 + C_1 l)^{\frac{3}{2}} \right\} + B_n J_{-\frac{1}{3}} \left\{ \frac{2\lambda_n}{3C_1} (C_0 + C_1 l)^{\frac{3}{2}} \right\} = 0,$$

and λ_n is a root of the equation

$$
\begin{vmatrix}
J_{\frac{1}{3}}\left(\frac{2\lambda_n}{3C_1}\,C_o^{\frac{3}{2}}\right), & J_{-\frac{1}{3}}\left(\frac{2\lambda_n}{3C_1}\,C_o^{\frac{3}{2}}\right) \\[2mm]
J_{\frac{1}{3}}\left\{\frac{2\lambda_n}{3C_1}\,(C_o+C_1 l)^{\frac{3}{2}}\right\}, & J_{-\frac{1}{3}}\left\{\frac{2\lambda_n}{3C_1}\,(C_o+C_1 l)^{\frac{3}{2}}\right\}
\end{vmatrix} = 0,
$$

in other words, if

$$
\frac{A_n}{B_n} = -\frac{J_{-\frac{1}{3}}(a\lambda_n)}{J_{\frac{1}{3}}(a\lambda_n)}\ ,
$$

and λ_n is a root of the equation

$$
J_{\frac{1}{3}}(a\lambda_n)\ J_{-\frac{1}{3}}(b\lambda_n) - J_{-\frac{1}{3}}(a\lambda_n)\ J_{\frac{1}{3}}(b\lambda_n) = 0,
$$

where

$$
a = \frac{2C_o^{\frac{3}{2}}}{3C_1}\ , \qquad b = \frac{2}{3C_1}\,(C_o+C_1 l)^{\frac{3}{2}}
$$

We thus see that the solution of the differential equation of forced vibration can be expressed in the form

$$
y = \cos(\beta t + \gamma) \sum_n \frac{u_n(x)}{\lambda_n^2 - \beta^2}\ \frac{\displaystyle\int_o^l F(x)\,u_n(x)\,dx}{\displaystyle\int_o^l (C_o+C_1 x)\,u_n^2(x)\,dx}
$$

where

$$
u_n(x) = A_n(C_o+C_1 x)^{\frac{1}{2}}\left[J_{-\frac{1}{3}}(a\lambda_n)J_{\frac{1}{3}}\left\{\frac{2\lambda_n}{3C_1}\,(C_o+C_1 x)^{\frac{3}{2}}\right\}\right.
$$

$$
\left. - J_{\frac{1}{3}}(a\lambda_n)\ J_{-\frac{1}{3}}\left\{\frac{2\lambda_n}{3C_1}\,(C_o+C_1 x)^{\frac{3}{2}}\right\}\right]
$$

We shall next consider a number of cases in which the density function is of the type

$$
\rho(x) = C + \epsilon\,\psi(x),
$$

where C is a constant quantity and ϵ is a very small quantity, that is to say, the density differs slightly from a constant value.

Example 2. As an example of this type, let us consider the case when

$$
\rho(x) = C_o + C_1\,e^{\delta x}.
$$

Prof. Karl Pearson[*] has shown that the solution of

$$\frac{d^2u}{dx^2} + \left(-n^2\beta^2 + a^2\beta^2 e^{2\beta x} \right) u = 0,$$

can be expressed in the form

$$u = \mathrm{AJ}_n \left(a e^{\beta x} \right) + \mathrm{BY}_n \left(a e^{\beta x} \right).$$

Thus, if we put ni for n, we see that the solution of

$$\frac{d^2u}{dx^2} + \lambda^2 \left(n^2\beta^2 + a^2\beta^2 e^{2\beta x} \right) u = 0,$$

is

$$u = \mathrm{AJ}_{ni} \left(a e^{\lambda\beta x} \right) + \mathrm{BY}_{ni}\left(a e^{\lambda\beta x} \right),$$

where $\mathrm{J}_{ni}(\theta)$ and $\mathrm{Y}_{ni}(\theta)$ are the solutions of the first and second kind of

$$\frac{d^2y}{d\theta^2} + \frac{1}{\theta}\frac{dy}{d\theta} + \left(1 + \frac{n^2}{\theta^2} \right) y = 0.$$

Using Schläfli's expression for the Bessel function of degree n (real or complex), *viz.*,

$$\mathrm{J}_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)\,d\theta - \frac{\sin n\pi}{\pi}\int_0^\infty e^{-n\theta - z\sinh\theta}\,d\theta.$$

we see that the solution of this equation can be written in the form

$$\mathrm{J}_{ni}(z) = \frac{1}{\pi}\int_0^\pi [\cosh n\theta \cos(z\sin\theta) + i \sinh n\theta \sin(z\sin\theta)]\,d\theta$$

$$\frac{i\sinh n\pi}{\pi}\int_0^\infty e^{-z\sinh\theta}(\cos n\theta - i\sin n\theta)\,d\theta.$$

In the present case, we upt

$$C_0 = n^2\beta^2, \quad C_1 = a^2\beta^2, \quad \delta = 2\beta.$$

Thus in this case the normal function is

$$\mathrm{A}_n\mathrm{J}_{ni}\left(ae^{\lambda_n\beta x} \right) + \mathrm{BY}_{ni}\left(ae^{\lambda_n\beta x} \right)$$

where $A_n : B$ is given by

$$A_n J_{n,i}(a) + B_n Y_{n,i}(a) = o ,$$
$$A_n J_{n,i}\left(ae^{\lambda_n \beta l}\right) + B_n Y_{n,i}\left(ae^{\lambda_n \beta l}\right) = o,$$

and λ_n is a root of the equation

$$J_{n,i}(a)Y_{n,i}\left(ae^{\lambda_n \beta l}\right) - Y_{n,i}(a)J_{n,i}\left(ae^{\lambda_n \beta l}\right) = o.$$

Example 3. As another example we shall consider the case, when

$$\rho(r) = \frac{C_1}{(a+\beta r)^2} + \frac{C}{(a+\beta r)^4}.$$

Prof. Karl Pearson* has shown that the solution of

$$\frac{d^2 u}{dr^2} + \left\{ \frac{A_0}{(a+r)^2} + \frac{A_1}{(a+r)^4} \right\} u = o$$

can be expressed in the form

$$u = \sqrt{(a+r)} \left\{ A J_{\sqrt{\left(\frac{1}{4} - \frac{A_0}{A_1}\right)}}\left(\frac{\sqrt{A_1}}{a+r}\right) + B Y_{\sqrt{\left(\frac{1}{4} - \frac{A_0}{A_1}\right)}}\left(\frac{\sqrt{A_1}}{a+r}\right) \right\}.$$

Putting

$$A_0 = \frac{\lambda^2 C_1}{\beta}, \quad A_1 = \frac{\lambda^2 C_2}{\beta}, \quad a\beta = a$$

we see that the solution of

$$\frac{d^2 u}{dx^2} + \lambda^2 \left\{ \frac{C_1}{(a+\beta r)^2} + \frac{C_2}{(a+\beta r)^4} \right\} u = o,$$

is

$$u = \sqrt{(a+\beta x)} \left\{ A J_{\sqrt{\left(\frac{1}{4} - \frac{C_1}{C_2}\right)}}\left(\lambda \cdot \frac{\sqrt{C_2}}{\sqrt{\beta}(a+\beta x)}\right) + B Y_{\sqrt{\left(\frac{1}{4} - \frac{C_1}{C_2}\right)}}\left(\lambda \cdot \frac{\sqrt{C_2}}{\sqrt{\beta}(a+\beta x)}\right) \right\}.$$

Hence the normal function for this case is

$$\sqrt{(a+\beta r)} \left\{ A_n J_{\sqrt{\left(\frac{1}{4} - \frac{C_1}{C_2}\right)}}\left(\lambda_n \cdot \frac{\sqrt{C_2}}{\sqrt{\beta}(a+\beta r)}\right) + B_n Y_{\sqrt{\left(\frac{1}{4} - \frac{C_1}{C_2}\right)}}\left(\lambda \cdot \frac{\sqrt{C_2}}{\sqrt{\beta}(a+\beta r)}\right) \right\},$$

provided $A_n : B_n$ is given by

$$A_n J_{\sqrt{\left(\frac{1}{4} - \frac{C_1}{C_2}\right)}}(a\lambda_n) + B_n Y_{\sqrt{\left(\frac{1}{4} - \frac{C_1}{C_2}\right)}}(a\lambda_n) = o$$

$$A_n J_{\sqrt{\left(\frac{1}{4}-\frac{C_1}{C_2}\right)}}(b\lambda_n) + B_n Y_{\sqrt{\left(\frac{1}{4}-\frac{C_1}{C_2}\right)}}(b\lambda_n) = 0$$

and λ_n is a root of the equation

$$J_{\sqrt{\left(\frac{1}{4}-\frac{C_1}{C_2}\right)}}(a\lambda_n) \; Y_{\sqrt{\left(\frac{1}{4}-\frac{C_1}{C_2}\right)}}(b\lambda_n) - J_{\sqrt{\left(\frac{1}{4}-\frac{C_1}{C_2}\right)}}(b\lambda_n) \; Y_{\sqrt{\left(\frac{1}{4}-\frac{C_1}{C_2}\right)}}(a\lambda_n) = 0,$$

where

$$a = \frac{\sqrt{C_2}}{a\sqrt{\beta}}, \qquad b = \frac{\sqrt{C_2}}{\sqrt{\beta}(a+\beta l)}.$$

Lastly, we shall consider a case where the normal function is not a Bessel-function but an associated Legendre function.

E ample 4. Suppose that the density function is given by

$$\rho'x) = C_0 + \frac{C_1}{\sin^2(c+x)},$$

and that the limit o to l is such that $\sin'c+x)$ does not vanish anywhere between the limits.

It is easily seen that the solution of

$$\frac{d^2 u}{dx^2} + \left[(n+\tfrac{1}{2})^2 - \frac{4m^2-1}{4\sin^2(c+x)} \right] u = 0,$$

is

$$u = \sqrt{\{\sin(c+x)\}}. \quad [A P_n^m(\mu) + B Q_n^m(\mu)],$$

where

$$\mu = \cos(c+x).$$

Hence this case can be discussed in exactly the same manner as the preceding ones on putting

$$C_0 = (n+\tfrac{1}{2})^2, \; C_1 = \frac{1-4m^2}{4}.$$

6. In physical applications, the impressed force is generally supposed to act in the immediate neighbourhood of one point $x=a$, and may usually be reckoned as a whole, so that writing

$$X = \int F(x)dx,$$

we see that if we suppose the force to be concentrated at the point $x=a$, the solution can be written in the form

$$y = \cos'\beta t + \gamma) \sum_n \frac{u_n(a) u_n(x)}{\lambda_n^2 - \beta^2} \cdot \frac{X}{\int_0^l \rho(x)u_n^2(x)dx}$$

Hence, as usual, all components vanish which have a node at the point where the force is supposed to be concentrated. It will also appear that this conclusion does not depend on any particular law of force or of density.

ON ROTATIONS ABOUT CONCURRENT AXES, AND THE POLAR OF A SPHERICAL POLYGEN

By C. E. CULLIS.

Summary. In Art. 1 it is shown that rotations of a rigid body about concurrent axes fixed in space can be replaced by the same rotations about the same axes moving with the body provided that the order in which the rotations are applied is reversed. In Art 2 it is shown that the polar of a spherical polygon admits of a unique definition and has the same uses as the polar of an ordinary spherical triangle. In Arts. 3 and 4 these results are used in discussing the complete generalisations of Rodrigues' and Sylvester's theorems regarding rotations.

1. Equivalent rotations about fixed and moving axes.

Throughout this paper we regard O as a fixed point of space, and all rotations as rotations of a given rigid body about axes drawn from O. An axis OA drawn from O will always be supposed to terminate in a point A lying on the sphere of unit radius whose centre is O. In every theorem the rotations are supposed to be either all right-handed or all left-handed, but each of them may be either positive or negative. A rotation through an angle a about an axis OA will be called simply a rotation a about OA. We may always consider a to be numerically less than 2π. The axis OA of a given rotation can always be so chosen that the angle a of the rotation is positive and not greater than π. The right-handed rotation $-a$ is the same as the right-handed rotation $2\pi-a$, and also the same as the left-handed rotation a.

The figures are drawn for the case in which O is at infinity and the unit radius is infinite, so that all rotations are about parallel axes perpendicular to the plane of the figure; but they must be regarded as figures drawn on the surface of the sphere of unit radius whose centre is O and looked at from a position outside the sphere, i.e. on the side remote from O.

Theorem I.—*Successive rotations of a rigid body through angles $a_1, a_2, \ldots a_{n-1}, a_n$ about the axes $OA_1, OA_2, \ldots OA_{n-1}, OA_n$ regarded as fixed in space are together equivalent to successive rotations through angles $a_n, a_{n-1}, \ldots a_2, a_1$ about the axes $OA_n, OA_{n-1}, \ldots OA_2, OA_1$ regarded as fixed in the body.*

A purely analytical proof of this theorem is given in *Matrices and Determinoids, Vol. II.* The proof given here is more elementary in character. The two cases considered include all possible cases.

CASE I.—*When there are only two rotations.*

The theorem will be proved in this case by showing that two succes-sive rotations a and β of a rigid body about two given axes OA and OB regarded as fixed in space are together equivalent to successive rotations β and a about the axes OB and OA regarded as fixed in the body.


Fig. 1.
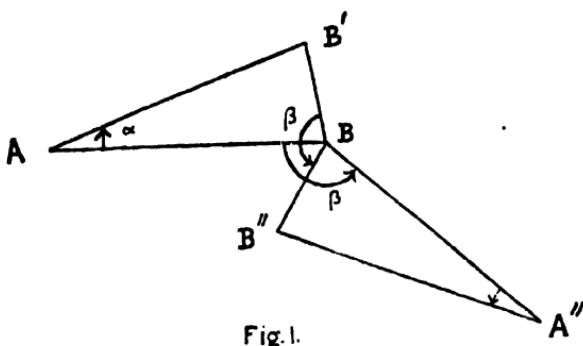
We suppose that a and β both lie between 0 and 2π, and that AB is the shorter great arc joining A and B. Let the rotation a about OA carry the point B to B', and let the rotation β about OB. carry the points A and B' to A'' and B'', so that the successive rotations a and β about OA and OB carry the triangle OAB to OA''B''. The theorem will be established if we show that the rotation β about OB followed by the rotation a about OA'' also carries the triangle OAB to OA''B''.

Now BAB' and B''A''B are two equal isosceles triangles whose vertical angles at A and A'' are both equal to a (or to $2\pi - a$). The rotation β about OB carries A to A'' leaving B unaltered, and the subsequent rotation a about OA'' carries B to B'' leaving A'' unaltered. Consequently the successive rotations β and a about OB and OA'' carry A, B to A'', B'', and therefore they also carry the triangle OAB to OA''B''.

If p and q are any two planes passing through an axis OL, and if θ is the angle of the rotation about OL which carries p to q, a reflexion in the plane p followed by a reflexion in the plane q is equivalent to a rotation 2θ about OL. When we make use of this fact, we have illustrations and more complete proofs of Case I of Theorem I in Figs. ii and iii, which are equivalent to Rodrigues' constructions for the resultant of two rotations. In these figures (which are drawn for right-handed rotations) every two triangles which have a side in common are the reflexions of one another in the plane through O and that common side.
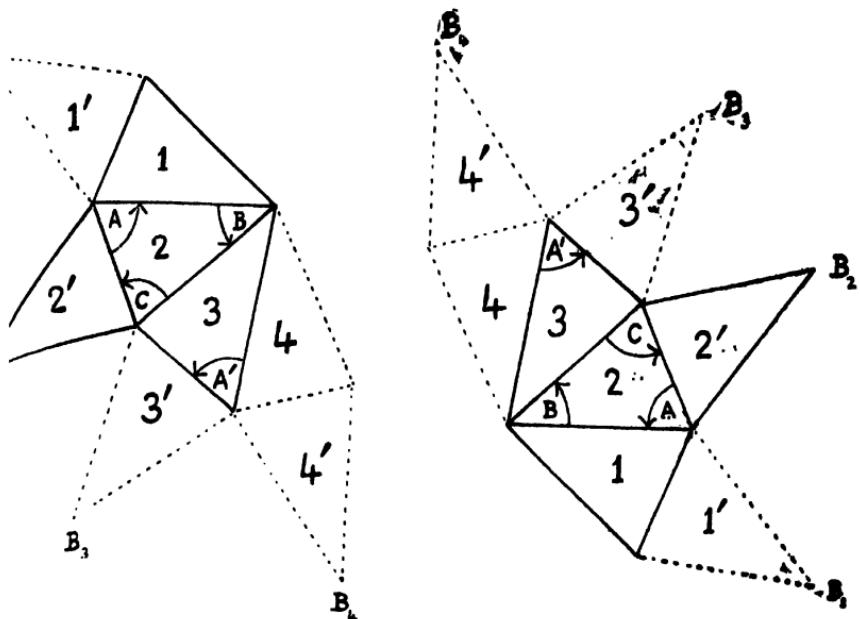
Fig. II.
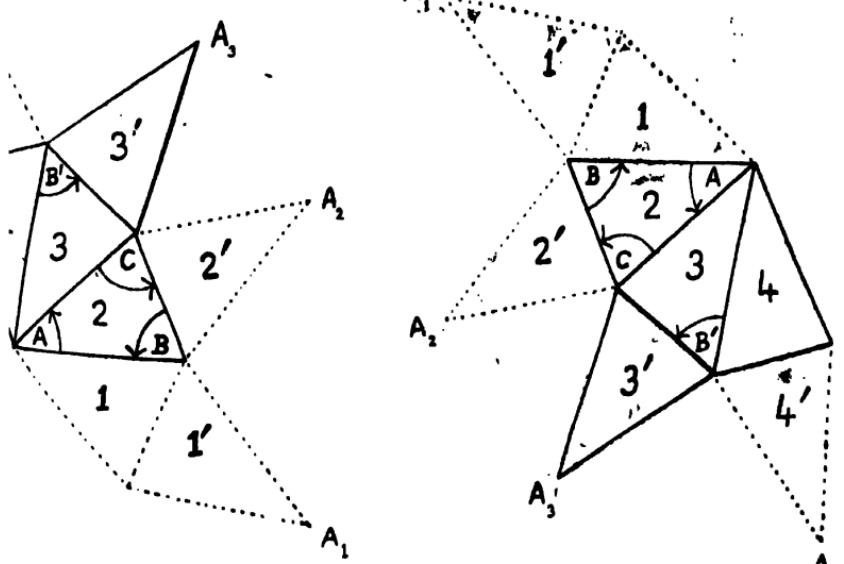
Fig. III.

To find directly the resultant of the two rotations a and β about OA and OB regarded as fixed in space, we construct on AB as base an ordinary spherical triangle ABC (left-handed when the rotations are right-handed, and right-handed when the rotations are left-handed) whose angles at A and B are $\frac{1}{2}a$ and $\frac{1}{2}\beta$, and then complete Fig. ii, in which the dotted triangles are superfluous. If A, B, C are the angles of the triangle ABC, we can write.

$$2A=a, \quad 2B=\beta, \quad 2C=\gamma.$$

The rotation a about OA carries $2'$ to 1, and the subsequent rotation β about OB carries 1 to 3. Thus these two successive rotations carry $2'$ to 3, and have as their resultant the single rotation $-\gamma$ about OC. Again the rotation β about OB carries $2'$, 2, A to $4'$, 4, A'; and the subsequent rotation a about A' carries $4'$ to 3. Thus the two successive rotations β and a about OB and OA' (*i.e.* about OB and OA regarded as moving with the body) also carry $2'$ to 3, and have $-\gamma$ about OC as their resultant. It will be observed however that the two rotations a and β about OA and OB carry $2'$ through 1 to 3, whereas the two rotations β and a about OB and OA' carry $2'$ through $4'$ to 3. Thus the paths from $2'$ to 3 are different in the two cases.

To find directly the resultant of the two rotations a and β about the axes OA and OB fixed in a moving body, we construct on AB as base an ordinary spherical triangle ABC (right-handed when the rotations are right-handed, and left-handed when the rotations are left-handed) whose angles at A and B are $\frac{1}{2}a$ and $\frac{1}{2}\beta$, and then complete Fig. iii, in which the dotted triangles are superfluous. The rotation a about OA carries 2 and B to 4 and B', and the subsequent rotation β about OB' carries 4 to $3'$. Thus the two successive rotations a and β about the moving axes OA and OB carry 2 through 4 to $3'$, and have as their resultant the single rotation $-\gamma$ about OC. Again the rotation β about OB carries 2 to $1'$, and the subsequent rotation a about OA carries $1'$ and 1 to $3'$ and 3. Thus the two successive rotations β and a about the fixed axes OB and OA carry 2 through $1'$ to $3'$, and also have as their resultant the rotation $-\gamma$ about OC.

CASE II.—*When there are more than two rotations.*

Taking r to be any positive integer greater than 2, we will make the hypothesis that the theorem is true for any number of rotations less than r (but not less than 2), and will show that it must then be true for r rotations through angles $a_1, a_2, \ldots a_r$ about the axes OA_1, $OA_2, \ldots OA_r$.

Let the rotation a_r about OA_r carry

$$A_1, A_2, \ldots A_r \qquad \text{to} \qquad A_1^{(1)}, A_2^{(1)}, \ldots A_r^{(1)};$$

let the rotation a_{r-1} about $OA_{r-1}^{(1)}$ carry

$$A_1^{(1)}, A_2^{(1)}, \ldots A_r^{(1)} \quad \text{to} \quad A_1^{(2)}, A_2^{(2)}, \ldots A_r^{(2)}$$

let the rotation a_2 about $OA_2^{(r-2)}$ carry

$$A_1^{(r-2)}, A_2^{(r-2)}, \ldots A_r^{(r-2)} \quad \text{to} \quad A_1^{(r-1)}, A_2^{(r-1)}, \quad A_r^{(r-1)}$$

and let the rotation a_1 about $OA_1^{(r-1)}$ carry

$$A_1^{(r-1)}, A_2^{(r-1)}, \ldots A_r^{(r-1)} \quad \text{to} \quad A_1^{(r)}, A_2^{(r)}, . A_r^{(r)}$$

so that $A_r^{(1)}, A_{r-1}^{(2)}, \ldots A_1^{(r)}$ are the same as $A_r, A_{r-1}^{(1)}, \cdots A_1^{(r-1)}$

Then we have to show that the successive rotations

$$a_1, a_2, \ldots a_r \quad \text{about} \quad OA_1, OA_2, \ldots OA_r \quad \ldots \quad (a)$$

are together equivalent to the successive rotations

$$a_r, a_{r-1} \quad \text{about} \quad OA_r, OA_{r-1}^{(1)}, \ldots OA_1^{(r-1)} . \quad (b)$$

Let the successive rotations

$$a_2, a_3, \ldots a_r \quad \text{about} \quad OA_2, OA_3, \ldots OA_r,$$

which by hypothesis have the same resultant as the successive rotations

$$a_r, a_{r-1}, \ldots a_2 \quad \text{about} \quad OA_r, OA_{r-1}^{(1)}, \ldots OA_2^{(r-2)},$$

be together equivalent to a rotation through an angle ϕ about the axis OI, so that the successive rotations (a) are together equilvalent to

the rotation a_1 about OA_1 followed by the rotation ϕ about OI. ... (a')

Since the rotation ϕ about OI carries A_1 to $A_1^{(r-1)}$, it follows from the hypothesis, or from Case I, that the successive rotations (a') are equivalent to

the rotation ϕ about OI followed by the rotation a_1, about $OA_1^{(r-1)}$, (b')

i.e., to the successive rotations (b); and this is what it was required to prove.

As an alternative proof let the successive rotations

$$a_1, a_2, \ldots a_{r-1} \quad \text{about} \quad OA_1, OA_2, \ldots OA_{r-1}$$

be together equivalent to a rotation through an angle ψ about OI, and let the rotation a_r about OA_r carry I to J. Then the successive rotations (a) are together equivalent to

the rotation ψ about OI followed by the rotation a_r about OA_r, ... (a'')

and by the hypothesis (or by Case I) these are equivalent to

the rotation a_r about OA_r followed by the rotation ψ about OJ. ... (b'')

Now since the rotation a_r about OA_r carries

$$A_1, A_2, \ldots A_{r-1}, I \quad \text{to} \quad A_1^{(1)}, A_2^{(1)}, \ldots A_{r-1}^{(1)}, J,$$

and the rotation ψ about OI is equivalent to the successive rotations

$$a_1, a_2, \ldots a_{r-1} \quad \text{about} \quad OA_1, OA_2, \ldots OA_{r-1},$$

the rotation ψ about OJ must be equivalent to the successive rotations

$$a_1, a_2, a_{r-1} \quad \text{about} \quad OA_1^{(1)}, OA_2^{(1)}, \ldots OA_{r-1}^{(1)},$$

and therefore by the hypothesis to the successive rotations

$$a_{r-1}, a_{r-2}, \ldots a_1 \quad \text{about} \quad OA_{r-1}^{(1)}, OA_{r-2}^{(2)}, \ldots OA_1^{(r-1)}.$$

Consequently the rotations (b'') must be equivalent to the rotations (b), *i.e.* the rotations (a) are equivalent to the rotations (b).

Fig. iv on p. 7 serves to illustrate Theorem I for four successive rotations through angles a, β, γ, δ about the axes OA, OB, OC, OD. The rotation a about OA carries the quadrilateral

$$0 \quad \text{to} \quad 1;$$

the rotation β about OB carries the quadrilaterals

$$0, 1 \quad \text{to} \quad 1', 2';$$

the rotation γ about OC carries the quadrilaterals

$$0, 1', 2' \quad \text{to} \quad 1'', 2'', 3'';$$

and the rotation δ about OD carries the quadrilaterals

$$0, 1'', 2'', 3'' \quad \text{to} \quad 1''', 2''', 3''', 4'''.$$

Thus the four successive rotations a, β, γ, δ about OA, OB, OC, OD carry the quadrilateral 0 to 4''', (through the intermediate positions 1, 2', 3'').

Now the rotation δ about OD carries the quadrilateral

$$0 \quad \text{to} \quad 1''';$$

the rotation γ about OC_1''' carries the quadrilateral

$$1''' \quad \text{to} \quad 2''';$$

the rotation β about OB_2''' carries the quadrilateral

$$2''' \quad \text{to} \quad 3''';$$

and the rotation a about OA_3''' carries the quadrilateral

$$3''' \quad \text{to} \quad 4'''.$$

Thus the rotations δ, γ, β, a about axes OD, OC, OB, OA regarded as fixed in a rigid body also carry the quadrilateral 0 to 4''', (but through the intermediate positions 1''', 2''', 3''').
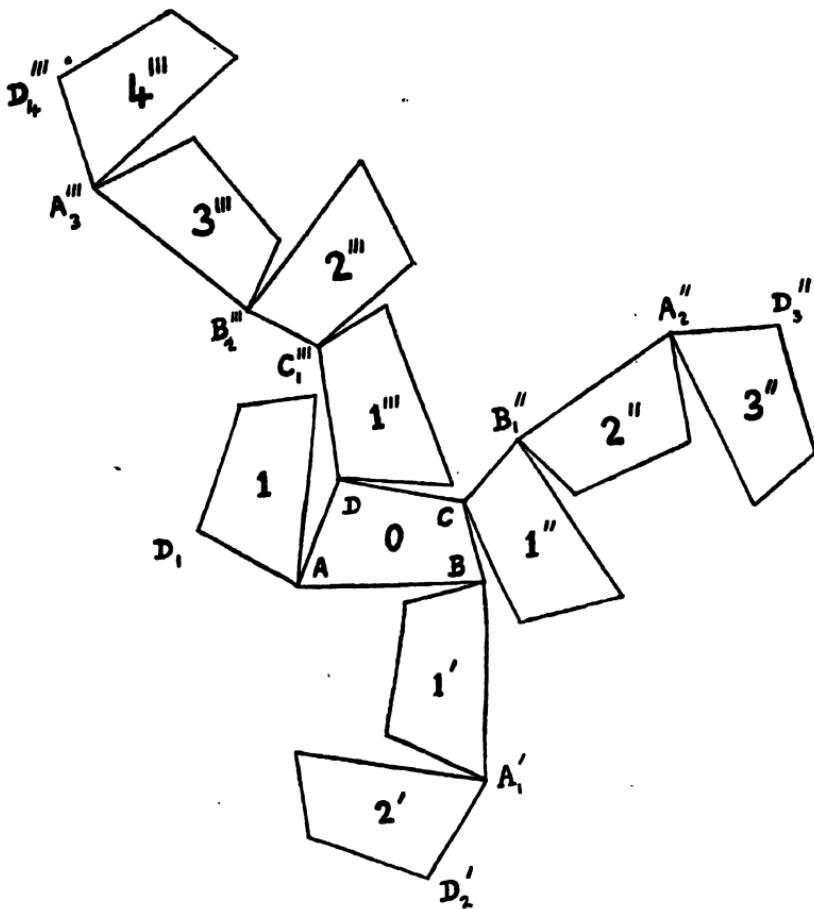
Fig. IV.

The figure also serves to illustrate the theorem for three successive rotations α, β, γ about OA, OB, OC, and for two successive rotations α, β about OA, OB.

If P is any point on the great circle in which the sphere of unit radius with O as centre is cut by the plane perpendicular to the axis OA, a rotation about OA which carries P to Q along an arc PQ of that great circle can be represented by the arc PQ, and will be called the rotation PQ. Again let PQ be any arc of a great circle on the sphere of unit radius with O as centre, and let A′ OA be the diameter of that sphere which is perpendicular to the plane of PQ. Then if OA is the right-handed axis of the rotation which carries P to Q along PQ, we call A the right-handed pole of the arc PQ, and

A' its left-handed pole. When rotations are, represented by arcs of great circles, Theorem I assumes the following form :

Successive rotations of a rigid body represented by arcs $A_1 B_1$, $A_2 B_2$, ... $A_{n-1} B_{n-1}$, $A_n B_n$ fi ed in space are together equivalent to successive rotations represented by arcs $A_n B_n$, $A_{n-1} B_{n-1}$, ... $A_2 B_2$, $A_1 B_1$ fixed in the body.

2. General definitions of a spherical polygon and its polar.

Let A, B, C, D, ... F, G, H be any points on the surface of the sphere of unit radius whose centre is O; and let $a=HA$, $b=AB$, $c=BC$, ... $g=FG$, $h=GH$ be arcs of great circles drawn from H to A, from A to B, from B to C, ... from F to G, from G to H. The figure HABCD ... FGH thus formed will be called a closed spherical polygon $\Omega = ABC \dots GH$ whose sides are a, b, c, ... g, h, and whose angular points are A, B, C, ... G, H. With the angular points A, B, C, ... G, H we associate angles A, B, C, ... G, H which we proceed to define in two different ways.

The angle between the two arcs of each of the pairs (AH, AB), (BA, BC), (CB, CD), ... regarded as essentially positive or signless quantities *not greater than* π will be denoted by a, β, γ, ... ; so that the positive quantity γ is the smallest angle of a rotation about OC (right-handed or left-handed) which will carry CB to CD or CD to CB·

We call Ω a *right-handed polygon* when, C being any angular point of Ω, the angle C is defined to be the angle *not numerically greater than* π of the *right-handed* rotation about OC which must be applied to the arc CD to make it lie along CB. We call Ω a *left-handed polygon* when, C being any angular point of Ω, the angle C is defined to be the angle *not numerically greater than* π of the *left-handed* rotation about OC which must be applied to the arc CD to make it lie along CB.

We can regard Ω either as a right-handed polygon or as a left-handed polygon. and it is not completely defined until one of these two aspects has been ascribed to it. When it is regarded as a right-handed polygon, we have $C=\gamma$ or $C=-\gamma$ according as the right-handed rotation or the left-handed rotation γ about OC carries CD to CB. When it is regarded as a left-handed polygon, we have $C=\gamma$ or $C=-\gamma$ according as the left-handed rotation or the right-handed rotation γ about OC carries CD to CB. Thus if A_1, B_1, C_1, ... are the angles at A, B, C, ... when we regard Ω as a right-handed polygon, and if A_2, B_2, C_2, ... are the angles at A, B, C, ... when we regard Ω as a left-handed polygon, we have

$$A_2 = -A_1, \ B_2 = -B_1, \ C_2 = -C_1, \dots .$$

The left-handed polygon HG .:. CBA has the same angles as the right-handed polygon ABC ... GH.

First let Ω=ABC...H be a closed *right-handed polygon*, and let A', B', C', D', ... H' be the *right-handed poles* of the arcs a=HA, b=AB, c=BC, d=CD, ... h=GH. Then if C is positive, the right-handed rotation $\pi-C=\pi-\gamma$ (which lies between 0 and π) carries C' to D' along an arc $c'=\pi-C$; and if C is negative, the right-handed rotation $\pi-C=\pi+\gamma$ (which lies between π and 2π) carries C' to D along an arc $c'=\pi-C$. Hence we can draw from A' to B', from B' to C' from C' to D' ... from H' to A', arcs a', b', c', ... h' such that

$$a'+A=\pi, \qquad b'+B=\pi, \qquad c'+C=\pi, \dots h'+H=\pi; \qquad \dots \ (1)$$

and then A, B, C, D, ... H are the right-handed poles of a', b', c', d', ... h'. The closed *right-handed polygon* Ω'=A' B' C' D' ... H' will be called the *right-handed polar* of Ω, or simply the *polar* of Ω. Since Ω is also the (right-handed) polar of Ω', the angles A', B', C', ... H' of Ω' are such that

$$a+A'=\pi, \qquad b+B'=\pi, \qquad c+C'=\pi, \dots h+H'=\pi. \qquad \dots \ (2)$$

The side a' is greater than π when and only when A is negative, and the angle A' is negative when and only when a is greater than π.

Next let Ω=ABC...HA be a closed *left-handed polygon*, and let A', B', C', D', ... H' be the *left-handed poles* of the arcs a=H A, b=A B, c=B C, d=C D, ... h=G H. Then if C is positive; the left-handed rotation $\pi-C=\pi-\gamma$ (which lies between 0 and π) carries C' to D' along an arc $c'=\pi-C$; and if C is negative, the left-handed rotation $\pi-C=\pi+\gamma$ (which lies between π and 2π carries C' to D' along an arc $c'=\pi-C$. Hence we can draw from A' to B', from B' to C', from C' to D', .. from H' to A' arcs a', b', c', ... h' given by the equations (1); and then A, B, C, D, ... H are the left-handed poles of a', b', c', d', ... h'. The closed *left-handed polygon* Ω'=A' B' C' D' ... H' will be called the *left-handed polar* of Ω, or simply the *polar* of Ω. Since Ω is also the (left-handed) polar of Ω', the angles A', B', C', ... H' of Ω' are given by the equations (2).

We can always without any ambiguity speak simply of the *polar* of a spherical polygon Ω=ABC ... H; it being the right-handed polar of Ω when Ω is a right-handed polygon, and being the left-handed polar of Ω when Ω is a left-handed polygon. The formal definitions of the angles of polygon given above have been introduced in order that this may be possible. The sides and angles of a closed polygon ABC ... H and its polar A' B' C' ... H' are always connected by the equations (1) and (2).

If Π is any open polygon formed with a number of successive sides and angles of Ω, its polar Π′ is the open polygon formed with the corresponding successive angles and sides of the polygon Ω′ polar to Ω.

We have hitherto considered polygons of the most general kind. When no two sides of a closed polygon $\Omega = ABC...H$, *i.e.* no two of the arcs HA, AB, BC, ... have a point in common, we will call Ω a *simple polygon*. A simple polygon ABC...H divides the whole surface of the sphere into two polygonal areas Ω_1 and Ω_2, each of which has the polygon as its sole boundary. If we imagine a person to walk on the outside of the sphere round the perimeter ABC ... H, passing through the angular points in this order, one of these areas (which we will suppose to be Ω_1) will lie constantly on his left, and the other area (which we will suppose to be Ω_2) will lie constantly on his right. We regard Ω_1 as the area of the right-handed polygon ABC...H, and Ω_2 as the area of the left-handed polygon ABC...H. Thus the area of a right-handed simple polygon ABC...H lies constantly on the left of a person describing its perimeter ABC...H in the manner just described, whilst the area of a left-handed simple polygon ABC...H lies constantly on his right. A simple closed polygon will often be identified with its polygonal area.

First let ABC...H be a right-handed simple closed polygon whose area is Ω_1; and let $A_1, B_1, C_1, ... H_1$ be the essentially positive or signless
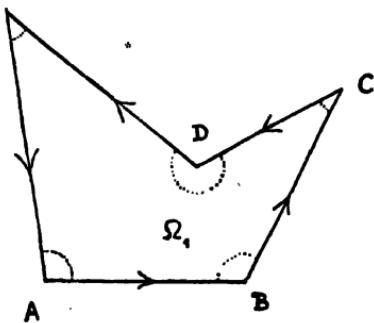
Fig. V.

interior angles of Ω_1 at A, B, C, ... H, each of which is less than 2π. If C_1 is less than π, we have

$$C_1 = \gamma, \qquad C = \gamma, \qquad C = C_1;$$

if C_1 is greater than π, we have

$$C_1 = 2\pi - \gamma, \quad C = -\gamma, \quad C = C_1 - 2\pi.$$

Thus C and C_1 can only differ by 0 or 2π, and the rotation C alway mean the same thing as the rotation C_1.

Next let ABC...H be a left-handed simple closed polygon whose area Ω_2; is and let A_2, B_2, C_2, ... H_2 be the essentially positive or
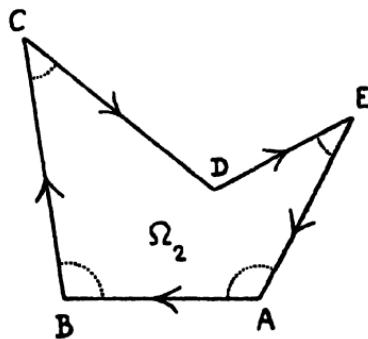
Fig. VI.

signless interior angles of Ω_2 at A, B, C, ... H, each of which is less than 2π. If C_2 is less than π, we have

$$C_2 = \gamma, \qquad C = \gamma, \qquad C = C_2 ;$$

if C_2 is greater than π, we have

$$C_2 = 2\pi - \gamma, \qquad C = -\gamma, \qquad C = C_2 - 2\pi.$$

Thus C and C_2 can only differ by 0 or 2π, and the rotation C always means the same thing as the rotation C_2.

A closed polygon ABC having three sides is a *spherical triangle* of the most general kind; and is necessarily a simple polygon. An *ordinary spherical triangle* is one whose sides and angles all lie between 0 and π. The term 'spherical triangle' is commonly understood to mean 'ordinary spherical, triangle.' Any three points A, B, C on the surface of the sphere of unit radius determine one and only one ordinary spherical triangle $\Omega = ABC$; its area Ω being the smaller of the two portions into which its perimeter divides the whole surface of the sphere, and its angles being the essentially positive interior angles of its area. An ordinary spherical triangle may be right-handed or left-handed, but it cannot have both aspects. Its polar is an ordinary spherical triangle having the same aspect.

The area of a general closed polygon can be defined to be an algebraical sum of areas of simple closed polygons, the areas of two corresponding right-handed and left-handed polygons formed with the same arcs together forming the whole surface of the sphere.

3. Rodrigues' theorems generalised.

Let $\Omega = ABC ... GH$ be any closed spherical polygon on the sphere of unit radius whose centre is O, and let its angles A, B, C, ... G, H

of the area Ω of the polygon. If we assign these new meanings to A, B, C, ... G, H, the truth of the theorems becomes evident when we construct on each side of Ω the reflexion of Ω in the plane passing through O and that side. We will illustrate this by considering the case of a simple spherical quadrilateral ABCD, the figures for which are given below.
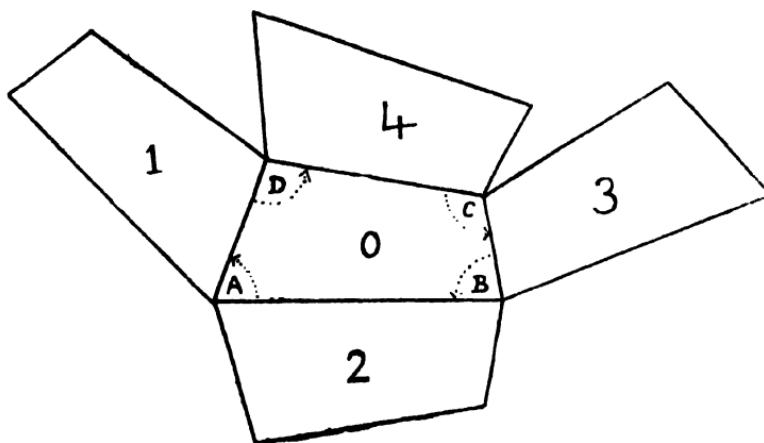
Fig. VII.
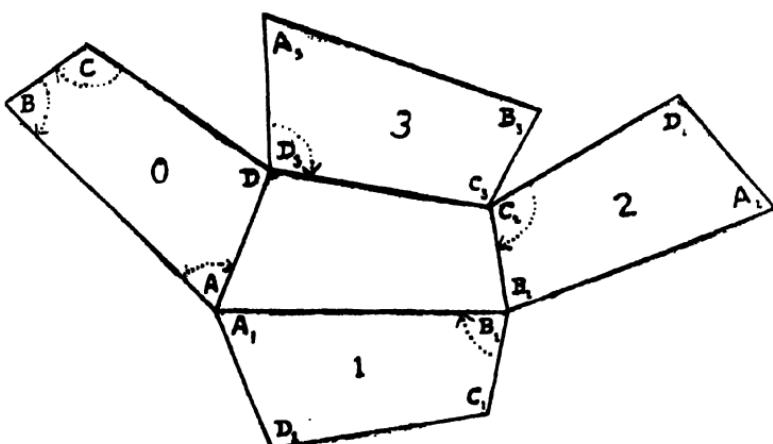
Fig. VIII.

To prove Theorem IIa directly we use the upper figure, which is constructed in the way just described. In this figure ABCD is a right-handed simple spherical quadrilateral whose area is marked 0; and the quadrilaterals 1, 2, 3, 4 are the reflexions of ABCD in the planes ODA, OAB, OBC, OCD. The right-handed rotation $-2A$ about OA carries

<div align="center">1   to   2;</div>

the subsequent right-handed rotation $-2B$ about OB carries

<div align="center">2   to   3;</div>

the subsequent right-handed rotation $-2C$ about OC carries

<div align="center">3   to   4;</div>

and the subsequent right-handed rotation $-2D$ about OD carries

<div align="center">4   to   1.</div>

Since these four successive rotations produce no resultant change in the position of the quadrilateral 1, they are together equivalent to a zero rotation.

To prove Theorem IIb directly we use the lower figure which is the same as the upper figure, but differently marked. In this figure ABCD is a left-handed simple quadrilateral whose area is marked 0. The left-handed rotation 2A about OA carries

<div align="center">0   to   1;</div>

the subsequent left-handed rotation 2B about OB_1 carries

<div align="center">1   to   2;</div>

the subsequent left-handed rotation 2C about OC_2 carries

<div align="center">2   to   3;</div>

and the subsequent left-handed rotation 2D about OD_3 carries

<div align="center">3   to   0.</div>

Thus the four successive left-handed rotations 2A, 2B, 2C, 2D about the axes OA, OB_1, OC_2, OD_3 (i.e. about the axes OA, OB, OC, OD regarded as fixed in the moving body) produce no resultant change in the position of the quadrilateral 0, and are therefore together equivalent to a zero rotation.

CASE II.—*When the polygon Ω is not simple.*

We will prove Theorem IIa for a right-handed polygon $\Omega = ABCD$ EFGHA which is not simple, and whose angles A, B, C ... are defined as in Art 2,

Fig. IX.
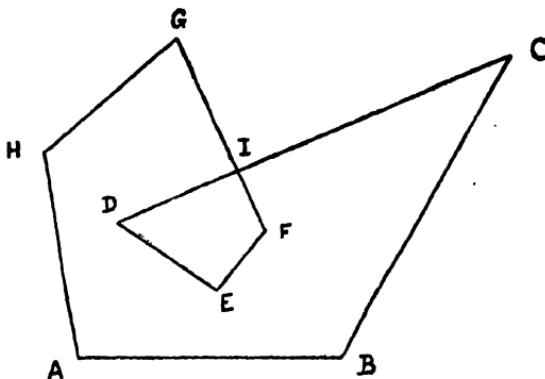
Let a point P starting from A describe the perimeter ABC...HA, let I in FG be the first point at which it crosses the path already described, let I be the intersection of FG with CD, and let I_1 and I_2 be the angles at A of the right-handed polygons IDEFI, ABCIGHA. Because I_1 and I_2 are the angles not numerically less than π of the right-handed rotations which respectively carry ID to IF and IG to IC, we have $I_2 = -I_1$.

Now IDEFI is a simple polygon for which the theorem is true. If then the theorem is also true for the right-handed polygon ABCIGHA, then (suppressing mention of the axes) each of the two sets of successive right-handed rotations

$$(-2D, -2E, -2F, -2I_1), \quad (-2I_2, -2G, -2H, -2A, -2B, -2C)$$

has a zero resultant. If we apply both sets of rotations in succession, the two successive rotations $-2I_1$, $-2I_2$ will cancel one another, and we see that the successive right-handed rotations

$$(-2D, -2E, -2F, -2G, -2H, -2A, -2B, -2C)$$

have a zero resultant, i.e. Theorem IIa is true for the polygon Ω.

Thus Theorem IIa is true for the polygon Ω if it is true for the right-handed polygon ABCIGHA which has fewer crossings. Treating this latter polygon in the same way, and repeating the process, we ultimately reduce the truth of the theorem for Ω to the truth of the theorem for simple polygons.

We conclude that Theorem IIa is true for all right-handed polygons, and by similar reasoning that it is true for all left-handed polygons.

We can prove Theorem IIb in the same way, or deduce it from Theorem IIa by the use of Theorem I,

4. Sylvester's theorems generalised.

Let $\Omega = ABC...GH$ be any closed spherical polygon on the sphere of unit radius whose centre is O. Then we have the following theorems:

Theorem IIIa.—If *the polygon* Ω *is fixed in space then:*

(1) *The resultant of the successive rotations.*

$$2AB, 2BC, 2CD \quad ... \quad 2GH, 2HA \qquad ... \quad (1)$$

applied to a rigid body is a zero rotation which leaves the position of the body unaltered

(2) *The resultant of the successive rotations*

$$2AB, 2BC, 2CD \quad ... \quad 2GH \qquad ... \quad (1')$$

is the rotation $2AH$.

Theorem IIIb.—If *the polygon* Ω *is fixed in a rigid body and moves with it, then:*

(1) *The resultant of the successive rotations*

$$2BA, 2CB, 2DC \quad ... \quad 2HG, 2AH \qquad ... \quad (2)$$

applied to the body is a zero rotation which leaves its position unaltered

(2) *The resultant of the successive rotations*

$$2BA, 2CB, 2DC \quad ... \quad 2HG \qquad ... \quad (2')$$

is the rotation $2HA$,

It is obvious that in (1) and (2) it must be possible to start with any one of the rotations, provided that the cyclical order of the successive rotations remains unaltered. This can be verified in the same way as the corresponding results in Theorems IIa and IIb.

The second part of each theorem is an immediate consequence of the first part. Also each of the two theorems follows at once from the other by the use of Theorem I. Consequently it is only necessary to prove the first part of one of the theorems.

Sylvester's theorems are Theorems IIIa and IIIb in the special case when Ω is an ordinary spherical triangle. In this special case the second parts of the theorems furnish constructions for the resultant of two arbitrary successive rotations of a rigid body represented by great arcs P_1Q_1, P_2Q_2 fixed in space or fixed in the moving body, the arcs being always replaceable by two arcs having a common extremity. The second parts of the theorems do not however furnish direct constructions for the resultant of more than two *arbitrarily given* rotations.

Theorems IIa and IIIa are mutually polar, each of them being deducible from the other by the properties of polar polygons when

the sides and angles of spherical polygons are defined as in Art. 2. To deduce Theorem IIIa from Theorem IIa let a, b, c, ...g, h and A, B, C,...G, H be the sides and angles of a closed right-handed polygon Ω=ABC...GH, and let A,'B,'C,'...G,'H' and $a,'b,'c,'...g,'h'$ be the corresponding angles and sides of the polar polygon Ω'=A'B'C' ...G'H', which is also right-handed. Then if we apply Theorem IIa to the polygon Ω', we see that the successive right-handed rotations

$$-2A', -2B', -2C',...-2H' \quad \text{about} \quad OA'.\ OB',\ OC',...OH'$$

have a zero resultant, *i.e.* by (1) and (2) of Art. 2, the successive right-handed rotations

$$2a, \quad 2b, \quad 2c, \quad ... \quad 2h \quad \text{about} \quad OA',\ OB',\ OC',...OH'$$

have a zero resultant. Since A', B', ... H' are the right-handed poles of the arcs a=HA, b=AB, c=BC, ... h=GH, this is Theorem IIIa for the polygon HAB ... FG. Thus Theorem IIIa is true for all polygons.

Similarly Theorems IIb and IIIb are mutually polar, and each of them can be deduced from the other by the properties of polar polygons. Consequently Theorem IIIb is true for all polygons.

We will now give direct proofs of Theorems IIIa and IIIb, considering two cases which include all possible cases.

CASE I.—*When Ω is a simple polygon.*

In this case the truth of the theorems becomes evident when we construct at each angular point of Ω the reflexion of Ω in the straight line joining O to that angular point. We will illustrate this by considering the case of a simple spherical quadrilateral ABCD.

To prove Theorem IIIa directly we use the upper figure on p. 19, in which ABCD is a quadrilateral whose area (when it is regarded as right-handed) is marked 0. The quadrilaterals 1, 2, 3, 4 are the reflexions of the quadrilateral 0 in OA, OB, OC, OD. The rotation 2AB carries

$$1 \quad \text{to} \quad 2.$$

the subsequent rotation 2BC carries

$$2 \quad \text{to} \quad 3,$$

the subsequent rotation 2CD carries

$$3 \quad \text{to} \quad 4,$$

and the subsequent rotation 2DA carries

$$4 \quad \text{to} \quad 1.$$

Since these four successive rotations produce no resultant change in the position of the quadrilateral 1, they are together equivalent to a zero rotation.
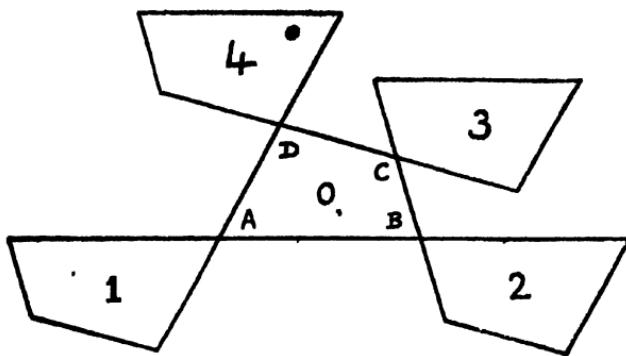
Fig. X.

- -
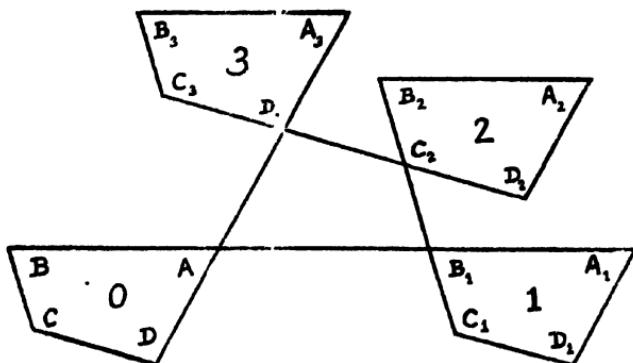
Fig. XI

To prove Theorem IIIb directly we use the lower figure, in which ABCD is a quadrilateral whose area (when it is regarded as right-handed) is marked 0. The figure is the same as before, but differently marked. The rotation 2BA carries

$$0 \quad \text{to} \quad 1.$$

the subsequent rotation $2C_1B_1$ carries

$$1 \quad \text{to} \quad 2$$

the subsequent rotation $2D_2C_2$ carries

<div align="center">2   to   3,</div>

and the subsequent rotation $2A_3D_3$ carries

<div align="center">3   to   0.</div>

Thus the four successive rotations $2BA$, $2C_1B_1$, $2D_2C_2$, $2A_3D_3$ (*i.e.* the rotations $2BA$, $2CB$, $2DC$, $2AD$ when we regard $ABCD$ as moving with the body) pr uce no resultant change in the position of the quadrilateral 0, and must be together equivalent to a zero rotation.

CASE II.—*When the polygon Ω is not simple.*

We will prove Theorem IIIa for the closed spherical polygon $\Omega =$ $ABCDEFGH$, and use the figure on p. 16. We suppose a point P starting from A to describe the perimeter ABC ... HA, and to cross the path already described for the first time at the point I on FG, I being the intersection of FG with CD. Then $IDEF$ is a simple closed polygon for which the theorem is true. If then the theorem is also true for the closed polygon $ABCIGH$, each of the two successive sets of rotations

<div align="center">(2ID, 2DE, 2EF, 2FI), (2IG, 2GH, 2HA, 2AB, 2BC, 2CI)</div>

has a zero resultant. If we apply both these sets of rotations in succession, the two successive rotations $2FI$, $2IG$ can be replaced by the single rotation $2FG$, and we see that the successive rotations

<div align="center">(2ID, 2DE, 2EF, 2FG, 2GH, 2HA, 2AB, 2BC, 2CI)</div>

have a zero resultant. If we prefix the rotation $2DI$ and postfix the rotation $2ID$, the resultant of the new set of rotations, being the same as that of $2DI$ and $2ID$, is again a zero rotation; and since we can now replace the last two rotations $2CI$ and $2ID$ by the single rotation $2CD$, we see that the successive rotations

<div align="center">(2DE, 2EF, 2FG, 2GH, 2HA, 2AB, 2BC, 2CD)</div>

have a zero resultant, *i.e.*, Theorem IIIa is true for the polygon Ω.

Thus Theorem IIIa is true for the polygon Ω if it is true for the closed polygon $ABCIGH$ which has fewer crossings than Ω. Treating the latter polygon in the same way, and repeating the process, we ultimately reduce the truth of the theorem for Ω to the truth of the theorem for simple polygons. We conclude that Theorem IIIa is true for all polygons.

We can prove Theorem IIIb in the same way, or consider it to be deduced from Theorem IIIa by the use of Theorem I.

On the Figures of Equilibrium of Two Rotating Masses of Fluid for the Exponential Potential $\dfrac{e^{-kr}}{r}$.

Part I.

By

Abanibhusan Datta.

§ 1. Introduction.

Although Newton's law of potential is universally admitted to be true for finite distances, there is considerable doubt as to its validity for small ones. The exponential potential $\dfrac{e^{-kr}}{r}$ seems to be a more reasonable expression to assume for such distances, for it has been shown by Neumann* that only under a potential of the form $\Sigma A \dfrac{e^{-kr}}{r}$ is the electric equilibrium possible inside a conductor.

In the present paper, I propose to study the figures of equilibrium of two rotating masses of fluid for this law of potential—a problem which has been investigated for the Newton's law by Sir George Darwin.†

It may be remarked that the exponential potential includes the Newtonian as a particular case and the Green's‡ potential

$$\sum_{n=1}^{n=\infty} A_n r^{(p_n-1)} \; ; \quad 0 \leq p_n < 1,$$

as a limiting case as is clear from the relation

$$r^{p-2} = \frac{1}{\Gamma(1-p)} \int_0^\infty \left(\frac{e^{-kr}}{r} \right) k^{-p} dk \quad , \; (0 \leq p < 1).$$

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* Neumann, Allgemeine untersuchungen über das Newton'sche Princip der Fernwirkungen, cap. 2. Teubner, Leipzig, 1896. (Quoted by Weatherburn).

† Darwin, *Phil. Trans.*, Vol. 178 A, (1887), pp. 379—428.

‡ Green, Mathematical investigations concerning the laws of equilibrium of fluids, etc., *Trans. Camb. Phil. Soc.*, 1833.

It is easy to see[*] that for this law of potential, Laplace's equation reduces to $(\nabla^2 - k^2)V = 0$ and the Poisson's equation to $(\nabla^2 - k^2)V = -4\pi\rho$.

The success in solving this problem is in a large measure due to a number of transformation formulae[†] with regard to the solutions of the equation of wave propagation in polar co-ordinates obtained by Dr. S. K. Banerji.

In Part II of this paper, I shall give the detailed numerical calculations of the results obtained by me in this paper, and also a number of diagrams showing the figures of equilibrium of the two rotating masses and shall also undertake a discussion of their stability.

I am grateful to Dr. S. K. Banerji for his having suggested this work to me and also for the helpful interest taken by him in the progress of this work.

§ 2.

We have first to find the potential of a homogeneous mass of liquid of unit density attracting according to Neumann's Law of Force $\dfrac{d}{dr}\dfrac{e^{-kr}}{r}$ whose free surface is approximately spherical.

Let the equation of the bounding surface be

$$r = a(1 + \Sigma a_n Y_n) \tag{1}$$

where Y_n is a spherical surface harmonic of degree n and a_n is a small quantity whose squares and products may be neglected.

Let V, V' be the potentials at an external and internal point respectively. Then it is easy to see that $(\nabla^2 - k^2)V = 0$ and $(\nabla^2 - k^2)V' = -4\pi$. It is obvious therefore that we can assume the following expressions for V and V' :—

$$V = \frac{4\pi e^{-kr}}{k^2 r}\left(a\cosh ak - \frac{\sinh ak}{k}\right) + \sum_{n=1}^{n=\infty} A_n \frac{K_{n+\frac{1}{2}}(kr)}{\sqrt{r}} Y_n \tag{2}$$

and $$V' = -\frac{4\pi \sinh kr}{k^2 r} e^{-ak}\left(a + \frac{1}{k}\right) + \sum_{n=1}^{n=\infty} B_n \frac{I_{n+\frac{1}{2}}(kr)}{\sqrt{r}} Y_n \tag{3}‡$$

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[*] Weatherburn, Green's functions for the equation $\nabla^2 u - k^2 u = 0$, etc., *Quart. Journ. of Math.*, Vol. XLVI (1915). See also *Phil. Mag.*, Vol. 30, Oct., 1915.

[†] Banerji, *Bulletin of Cal. Math. Soc.*, Vols. IV and V.

[‡] The expression for the potential at an internal point of a sphere for Neumann's law is

$$-\frac{4\pi \sinh kr}{k} e^{-ak}\left(a + \frac{1}{k}\right) + \frac{4\pi}{k^2}.$$

The constant term $\dfrac{4\pi}{k^2}$ has been neglected here.

where

$$K_n(x) = \frac{\pi}{2 \sin n\pi} e^{-\frac{1}{2}ni\pi} \left[J_{-n}(i x) - e^{ni\pi} J_n(i x) \right]$$

and

$$J_n(i x) = I_n(x),$$

and the constants A_n's and B_n's have to be determined by means of the two following conditions to be satisfied at the surface of the liquid mass :—

$$\left. \begin{array}{l} V = V' + \text{constant} \\[2mm] \dfrac{dV}{dr} = \dfrac{dV'}{dr} \end{array} \right\} \quad \text{when } r = a\,(1 + \textstyle\sum a_n Y_n). \qquad (4)$$

Since A_n's and B_n's are small quantities of the order a, we may in the small terms put $r = a$ but in the first term, we must give to r its full value from (1). We easily find from (4) that

$$A_n K_{n+\frac{1}{2}}(ak) = B_n\, I_{n+\frac{1}{2}}(ak), \qquad (5)$$

$$\frac{4\pi}{k^2}\left(a\cosh ak - \frac{\sinh ak}{k}\right) a_n \frac{e^{-ak}}{a^2}\,(k^2 a^2 + 2ak + 2) + A_n \frac{1}{a^{\frac{1}{2}}} \frac{d}{da} K_{n+\frac{1}{2}}(ak)$$

$$= -\frac{4\pi}{k^2} e^{-ak}\left(a + \frac{1}{k}\right) a_n \frac{\sinh ak\,(k^2 a^2 + 2) - 2ak\cosh ak}{a^2}$$
$$+ B_n \frac{1}{\sqrt{a}}\frac{d}{da} I_{n+\frac{1}{2}}(ak) \qquad (6)$$

whence

$$A_n = -\frac{4\pi a^{\frac{3}{2}} a_n\, I_{n+\frac{1}{2}}(ak)}{I_{n+\frac{1}{2}}(ak)\dfrac{d}{da} K_{n+\frac{1}{2}}(ak) - K_{n+\frac{1}{2}}(ak)\dfrac{d}{da} I_{n+\frac{1}{2}}(ak)}$$

$$= -4\pi a^{\frac{3}{2}} a_n f_{n+\frac{1}{2}}(ak),$$

where

$$f_{n+\frac{1}{2}}(ak) = \frac{I_{n+\frac{1}{2}}(ak)}{I_{n+\frac{1}{2}}(ak)\dfrac{d}{da} K_{n+\frac{1}{2}}(ak) - K_{n+\frac{1}{2}}(ak)\dfrac{d}{da} I_{n+\frac{1}{2}}(ak)}$$

Hence

$$V = \frac{4\pi e^{-kr}}{k^2 r}\left(a\cosh ak - \frac{\sinh ak}{k}\right) - 4\pi a^{\frac{3}{2}} \sum_{n=1}^{n=\infty} a_n f_{n+\frac{1}{2}}(ak)$$

$$\frac{K_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\, Y_n \qquad (7)$$

§ 3.

Let us now suppose that there are two masses of liquid whose free surfaces are approximately spheres and whose centres are o and O.

Let there be two sets of rectangular axes whose origins are o and O respectively and let the axis of z be measured from o to O and of Z from O to o. Let $Oo=c$; let the axis of rotation be a line paralled to ox drawn through some point on Oo whose distances from o and O are d and D respectively.
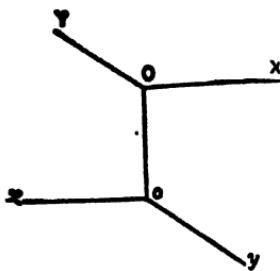
If Ω be the potential of the centrifugal forces, we have

$$\Omega = \tfrac{1}{2}\omega^2 \, (y^2 + z^2 + d^2 - 2dz)$$

Now, if r, $\cos^{-1}\mu$, ϕ, be the polar co-ordinates referred to o as origin

$$p_1(\mu) = \mu, \; p_2(\mu) = \tfrac{1}{2}(3\mu^2 - 1), \; p_2{}^2(\mu) = 3(1-\mu^2),$$

whence

$$\Omega = \tfrac{1}{2}\omega^2 \, (d^2 - 2drp_1 + \tfrac{1}{3}r^2 p_2 + \tfrac{2}{3}r^2 - \tfrac{1}{6}r^2 p_2{}^2 \cos 2\phi), \tag{8}$$

Similarly, the value of Ω referred to the other origin O is

$$\Omega = \tfrac{1}{2}\omega^2(D^2 - 2DRP_1 + \tfrac{1}{3}R^2 P_2 + \tfrac{2}{3}R^2 - \tfrac{1}{6}R^2 P_2{}^2 \cos 2\phi) \tag{9}$$

§ 4.

It will be hereafter necessary to employ certain transformation formulæ with regard to Bessel Functions. These formulæ have been given by Dr. S. K. Banerji in a paper* published by him in the Bulletin of the Calcutta Mathematical Society, Vol. V. The formulae which will be used in this paper are quoted below :—

$$\frac{K_{n+\frac{1}{2}}(ik'R)}{\sqrt{R}} \, P_n(\mu) = \sum_{p=0}^{p=\infty} A_{n,\,p} \, \frac{J_{p+\frac{1}{2}}(k'r)}{\sqrt{r}} \, p_p(\mu), \; if \; r < c \tag{10}$$

where

$$A_{n,\,p} = \frac{2p+1}{2}(-1)^n \, i^p \, \underset{a=0}{\mathrm{Lim}} \int_{-1}^{1} P_n(z) \, P_p(z) \, \phi\,(a, c, z) \, dz$$

* S. K. Banerji, "On Electromagnetic Waves due to Electrical Oscillations on the surface of a thin spherical shell in the presence of a non-concentric conducting Sphere". *Bulletin Cal. Math. Soc.* Vol. V. p. 21, (1913-14).

See also a paper by Banerji in Vol. IV of the Bulletin.

where

$$\phi(a, c, s) = \sqrt{\frac{2\pi}{kc}} \sum_{s=o}^{s=\infty} (-1)^s \, e^{-s^2a} (s+\tfrac{1}{2}) \, K_{s+\frac{1}{2}}(ik'c) \, P_s(s).$$

$$\frac{K_{s+\frac{1}{2}}(ik'R)}{\sqrt{R}} \, P_s^m(\mu) \cos m\phi = \sum_{p=o}^{p=\infty} A_{s, p, m} \frac{J_{p+\frac{1}{2}}(k'r)}{\sqrt{r}} \, p_p^m(\mu) \cos m\phi \tag{11}$$

if $r < c$, where

$$A_{s, p, m} = \frac{2p+1}{2} \frac{(p-m)!}{(p+m)!} (-1)^s \, i^p \, \underset{a=o}{\mathrm{Lim}} \int_{-1}^{1} P_p^m(z) P_s^m(z) \phi(a, c, s) \, dz.$$

The following transformation formula which is well-known will also be used in this paper,

$$\frac{e^{-kR}}{R} = \frac{2}{\sqrt{cr}} \sum_{p=}^{p=\infty} i^{s\,p+\frac{3}{2}} (p+\tfrac{1}{2}) \, I_{p+\frac{1}{2}}(kr) \, K_{p+\frac{1}{2}}(kc) \, p_p(\mu) \tag{A}$$

§ 5.

Let V, v be the potentials at an external point of the solids O, o respectively. The condition that the free surfaces of the two masses of liquid should be equipotential surfaces is that the equation

$$V + v + \Omega = \text{constant,}$$

should be satisfied at each of the free surfaces. Since the free surfaces are approximately spherical, v will consist of a series of harmonics of the form (7) referred to the origin o and V of a similar series referred to O. In order that the above condition may be satisfied, it is necessary that when $V + v$ is transferred by means of (10) and (11) into two separate series of harmonics referred to the two origins O and o respectively, the co-efficients of all the harmonics must vanish except those of $P_1(\mu)$, $p_1(\mu)$, $P_s(\mu)$, $p_s(\mu)$, $P_s^2(\mu) \cos 2\phi$, $p_s^2(\mu) \cos 2\phi$ and also the co-efficients of $P_1(\mu)$, $p_1(\mu)$, $P_s(\mu)$, $p_s(\mu)$, $P_s^2(\mu) \cos 2\phi$, $p_s^2(\mu) \cos 2\phi$ must be determined so as to annul those terms involving these quantities in Ω.

The terms R^2 and r^2 need not be considered for since the corresponding forces are symmetrical about each origin, they produce no departure from sphericity.

§ 6.

We shall now consider the potential $V + v$. Let the equations of the two surfaces be

$$\frac{r}{a} = 1 + \sum_{n=2}^{n=\infty} a_n p_n(\mu) + \sum_{n=2}^{n=\infty} a'_n p_n^2(\mu) \cos 2\phi \quad \dots \quad \dots \quad (12)$$

$$\frac{R}{A} = 1 + \sum_{n=2}^{n=\infty} \beta_n P_n(\mu) + \sum_{n=2}^{n=\infty} \beta'_n P_n^2(\mu) \cos 2\phi \dots \quad \dots \quad (13)$$

where a's, a''s, β's and β''s are unknown co-efficients whose values are to be determined.

It appears from (7) that

$$v = \frac{4\pi e^{-hr}}{h^2 r} \left(a \cosh ak - \frac{\sinh ak}{h} \right)$$

$$- 4\pi a^{\frac{3}{2}} \sum_{n=2}^{n=\infty} a_n f_{n+\frac{1}{2}}(ak) \frac{K_{n+\frac{1}{2}}(rh)}{\sqrt{r}} - p_n(\mu)$$

$$- 4\pi a^{\frac{1}{2}} \sum_{n=2}^{n=\infty} a'_n f_{n+\frac{1}{2}}(ak) \frac{K_{n+\frac{1}{2}}(kr)}{\sqrt{r}} p_n^2(\mu) \cos 2\phi \qquad (14)$$

$$V = \frac{4\pi e^{-kR}}{Rk^2} \left(A \cosh Ah - \frac{\sinh Ak}{k} \right)$$

$$- 4\pi A^{\frac{1}{2}} \sum_{n=2}^{n=\infty} \beta_n f_{n+\frac{1}{2}}(Ak) \frac{K_{n+\frac{1}{2}}(kR)}{\sqrt{R}} P_n(\mu)$$

$$- 4\pi A^{\frac{1}{2}} \sum_{n=2}^{n=\infty} \beta'_n f_{n+\frac{1}{2}}(Ak) \frac{K_{n+\frac{1}{2}}(kR)}{\sqrt{R}} P_n^2(\mu) \cos 2\phi \qquad (15)$$

Putting $ik' = k$ in (10) and (11) and transferring V to o by the resulting formulae and formula (A), we obtain

$$V + v = \frac{4\pi e^{-kr}}{^2 r} \left(a \cosh ak - \frac{\sinh ak}{k} \right)$$

$$-4\pi a^{\frac{3}{2}} \sum_{n=2}^{n=\infty} a_n f_{n+\frac{1}{2}}(ak) \frac{K_{n+\frac{1}{2}}(kr)}{\sqrt{r}} p_n(\mu)$$

$$-4\pi a^{\frac{3}{2}} \sum_{n=2}^{n=\infty} a_n' f_{n+\frac{1}{2}}(ak) \frac{K_{n+\frac{1}{2}}(kr)}{\sqrt{r}} p_n^2(\mu) \cos 2\phi$$

$$+ \frac{4\pi}{k^2} \left(A \cosh Ak - \frac{\sinh Ak}{k} \right) \frac{2}{\sqrt{cr}}$$

$$\sum_{p=0}^{p=\infty} i^{3p+\frac{3}{2}} (p+\frac{1}{2}) I_{p+\frac{1}{2}}(kr) K_{p+\frac{1}{2}}(kc) p_p(\mu)$$

$$- 4\pi A^{\frac{3}{2}} \sum_{n=2}^{n=\infty} \beta_n f_{n+\frac{1}{2}}(Ak) \sum_{p=0}^{p=\infty} A_{n, p} \frac{I_{p+\frac{1}{2}}(kr)}{\sqrt{r}} P_p(\mu)$$

$$- 4\pi A^{\frac{3}{2}} \sum_{n=2}^{n=\infty} \beta_n' f_{n+\frac{1}{2}}(Ak) \sum_{p=0}^{p=\infty} A_{n, p, 2} \frac{I_{p+\frac{1}{2}}(kr)}{\sqrt{r}}$$

$$p_p^2(\mu) \cos 2\phi \tag{16}$$

This quantity is to vanish when r has the value given by (12) with the exception of the terms involving $p_1(\mu)$, $p_2(\mu)$, $p_2^2(\mu) \cos 2\phi$.

Since the squares and products of small quantities are to be neglected, we may put $r = a$ in all the terms except the first in which we must give to r its full value provided that kc is fairly large.

Hence equating the co-efficients of $p_m(\mu)$, m being not equal to 1 or 2, we obtain

$$- \frac{4\pi}{k^2} \left(a \cosh ak - \frac{\sinh ak}{k} \right) \frac{e^{-ak}}{a} (ka + 1) a_m$$

$$-4\pi a^{\frac{3}{2}} f_{m+\frac{1}{2}}(ak) \frac{K_{m+\frac{1}{2}}(ak)}{\sqrt{a}} a_m$$

$$+\frac{4\pi}{k^2}\left(A\cosh Ak-\frac{\sinh Ak}{k}\right)\frac{2}{\sqrt{ca}}\;i^{\,3m+\frac{3}{2}}\;(m+\tfrac{1}{2})\;I_{m+\frac{1}{2}}\,(ak)\;K_{m+\frac{1}{2}}\,(kc)$$

$$-4\pi A\;\overset{\frac{3}{2}\;n\,=\,\infty}{\underset{n\,=\,2}{\sum}}\;\beta_n\,f_{n+\frac{1}{2}}\,(Ak)\;A_{n,\,m}\;\frac{I_{m+\frac{1}{2}}\,(ah)}{\sqrt{a}}\;=0.$$

Therefore $a_m=$

$$\frac{1}{k^2}\left(A\cosh Ak-\frac{\sinh Ak}{k}\right)\frac{2}{\sqrt{ca}}\;i^{\,3m+\frac{3}{2}}(m+\tfrac{1}{2})\;I_{m+\frac{1}{2}}\,(nk)\;K_{m+\frac{1}{2}}\,(kc)$$

$$\left(a\cosh ak-\frac{\sinh ak}{k}\right)\frac{e^{-ak}}{k^2 a}\;(ka+1)\;+\;af_{m+\frac{1}{2}}\,(ak)\;K_{m+\frac{1}{2}}\,(ak)$$

$$A\;\overset{\frac{3}{2}\;n\,=\,\infty}{\underset{n\,=\,2}{\sum}}\;\beta_n\,f_{n+\frac{1}{2}}\,(Ak)\;A_{n,\,m}\;I_{m+\frac{1}{3}}\,(ak)\;1\,\sqrt{a}$$

$$a\cosh ak-\frac{\sinh ak}{k}\Big)\frac{e^{-ak}}{k^2 a}\;(ka+1)\;+\;a\,f_{m+\frac{1}{2}}\,(ak)\;K_{m+\frac{1}{2}}\,(ak)$$

For the purpose of obtaining an approximate solution, it will be sufficient to retain only the first term in this expression and we thus obtain

$$a_m=\frac{\dfrac{1}{k^2}\left(A\cosh Ak-\dfrac{\sinh Ak}{k}\right)\dfrac{2}{\sqrt{ca}}\;i^{\,3m+\frac{3}{2}}\,(m+\tfrac{1}{2})\;I_{m+\frac{1}{2}}\,(ak)\;K_{m+\frac{1}{2}}\,(kc)}{\left(a\cosh ak-\dfrac{\sinh ak}{k}\right)\dfrac{e^{-ak}}{k^2 a}(ka+1)\;+\;af_{m+\frac{1}{2}}\,(ak)\;K_{m+\frac{1}{2}}\,(ak)}$$

$$\tag{17}$$

By symmetry

$$\beta_m=\frac{\dfrac{1}{k^2}\,(a\cosh ak-\sinh ak)\dfrac{2}{\sqrt{cA}}\;i^{\,3m+\frac{3}{2}}\,(m+\tfrac{1}{2})\;I_{m+\frac{1}{2}}\,(Ak)\;K_{m+\frac{1}{2}}\,(kc)}{\left(A\cosh Ak-\dfrac{\sinh Ak}{k}\right)\dfrac{e^{-Ak}}{k^2 A}\,(kA+1)\;+\;Af_{m+\frac{1}{2}}\,(Ak)\;K_{m+\frac{1}{2}}\,(Ak)}$$

$$\tag{18}$$

§ 7.

In order to determine the angular velocity we have to equate to zero the sum of the zonal harmonic terms of the first degree in Ω and in (16).

We thus obtain by giving to r its full value from (12).

$$\frac{4\pi}{k^2}\left(A\cosh Ak - \frac{\sinh Ak}{k}\right)\frac{3}{\sqrt{ca}}\ i^{\frac{1}{2}}\ I_{\frac{3}{2}}\ (ak)\ \ K_{\frac{1}{2}}\ (kc)$$

$$-4\pi A^{\frac{3}{2}}\sum_{n=2}^{n=\infty}\ \beta_n\, f_{n+\frac{1}{2}}\,(Ak)\ A_{n,1}\ \frac{I_{\frac{1}{2}}\,(Ak)}{\sqrt{a}} - \omega^2 ad = 0$$

Similarly

$$\frac{4\pi}{k^2}\left(a\cosh ak - \frac{\sinh ak}{k}\right)\frac{3}{\sqrt{cA}}\ i^{\frac{1}{2}}\ I_{\frac{3}{2}}\,(Ak)\ K_{\frac{1}{2}}\,(kc)$$

$$-4\pi a^{\frac{3}{2}}\sum_{n=2}^{n=\infty}\ a_n f_{n+\frac{1}{2}}\,(ak)\ A_{n,1}\ \frac{I_{\frac{1}{2}}\,(Ak)}{\sqrt{A}} - \omega^2 AD = 0$$

Hence adding these two expressions and remembering that
$$D + d = c,$$

we get

$$\omega^2 = \frac{12\pi}{k^2\, c^{\frac{3}{2}}}i^{\frac{1}{2}}\ K_{\frac{1}{2}}\,(kc)\left\{\left(A\cosh Ak - \frac{\sinh Ak}{K}\right)\frac{I_{\frac{3}{2}}(ak)}{a^{\frac{3}{2}}}\right.$$

$$\left.\cdot\left(a\cosh ak - \frac{\sinh ak}{k}\right)\frac{I_{\frac{3}{2}}(Ak)}{A^{\frac{3}{2}}}\right\}\quad (19)$$

§ 8.

Equating to zero the sum of the co-efficients of $p_2\,(\mu)$ in (16) and the co-efficient of $p_2\,(\mu)$ in Ω, after giving to r its value, we get

$$-\frac{4\pi}{k^2}\left(a\cosh ak - \frac{\sinh ak}{k}\right)\,(ka+1)\,\frac{e^{-ak}}{a}a_2 - 4\pi\,a\,f_{\frac{3}{2}}\,(ak)\ K_{\frac{3}{2}}(ak)\ a_2$$

$$+\frac{4\pi i^{\frac{1}{2}}}{k^2}\left(A\cosh Ak - \frac{\sinh Ak}{k}\right)\frac{5}{\sqrt{c}}\ I_{\frac{5}{2}}\,(ak)\ K_{\frac{5}{2}}\,(kc)$$

$$-4\pi A^{\frac{3}{2}}\sum_{n=2}^{n=\infty}\ \beta_n\ f_{n+\frac{1}{2}}\,(Ak)\ A_{n,2}\ \frac{I_{\frac{3}{2}}\,(ak)}{\cdot} + \frac{1}{2}\,a^2\,\omega^2 = 0$$

Neglecting smaller terms we get

$$a_2 = \frac{\frac{20\pi}{k^2\sqrt{c}}\ i^{-\frac{1}{2}}\left(A\cosh Ak - \frac{\sinh Ak}{k}\right)I_{\frac{5}{2}}(ak)\,K_{\frac{5}{2}}(kc) + \frac{1}{6}a^2\,\omega^2}{4\pi\left\{\frac{1}{k^2}\left(a\cosh ak - \frac{\sinh ak}{k}\right)\frac{e^{-ak}}{a}(ka+1) + a\,f_{\frac{5}{2}}(ak)\,K_{\frac{5}{2}}(ak)\right\}}$$

(20)

From symmetry,

$$\beta_2 = \frac{\frac{20\pi}{k^2\sqrt{c}}\ i^{-\frac{1}{2}}\left(a\cosh ak - \frac{\sinh ak}{k}\right)I_{\frac{5}{2}}(Ak)\,K_{\frac{5}{2}}(kc) + \frac{1}{6}A^2\,\omega^2}{4\pi\left\{\frac{1}{k^2}\left(A\cosh Ak - \frac{\sinh Ak}{k}\right)\frac{e^{-Ak}}{A}(kA+1) + A\,f_{\frac{5}{2}}(kA)\,K_{\frac{5}{2}}(Ak)\right\}}$$

(21)

where ω^2 is given by (19).

§ 9.

Equating the sum of the co-efficients of the harmonics $p_2{}^2(\mu)\cos 2\phi$ in (16) to that in Ω and giving to r its value, we get

$$-\frac{4\pi}{k^2}\left(a\cosh ak - \frac{\sinh ak}{k}\right)\frac{e^{-ak}}{a}(ka+1)\ a_2' - a_2'\ 4\pi a\,f_{\frac{5}{2}}(ak)K_{\frac{5}{2}}(ak)$$

$$-4\pi A^{\frac{3}{2}}\sum_{n=2}^{n=\infty}\beta'_n\,f_{n+\frac{1}{2}}(Ak)\,A_{n,\,2,\,2}\frac{I_{\frac{5}{2}}(ak)}{\sqrt{a}} - \frac{1}{12}a^2\,\omega^2 = 0.$$

Neglecting smaller terms, we get

$$a_2' = -\frac{\frac{1}{12}a^2\,\omega^2}{4\pi\left\{\frac{1}{k^2}\left(a\cosh ak - \frac{\sinh ak}{k}\right)\frac{e^{-ak}}{a}(ka+1) + a\,f_{\frac{5}{2}}(ak)\,K_{\frac{5}{2}}(ak)\right\}}$$

(22)

From symmetry

$$\beta_2' = -\frac{\frac{1}{12}A^2\,\omega^2}{4\pi\left\{\frac{1}{k^2}\left(A\cosh Ak - \frac{\sinh Ak}{k}\right)\frac{e^{-Ak}}{A}(kA+1) + A\,f_{\frac{5}{2}}(Ak)\,K_{\frac{5}{2}}(Ak)\right\}}$$

(23)

where ω^2 is given by (19).

§ 10.

Equating to zero, the co-efficients of p_m^2 (μ) $\cos 2\phi$, $m \neq 2$, we obtain

$$-\frac{4\pi}{k^2}\left(a\cosh ak - \frac{\sinh ak}{k}\right)\frac{e^{-ak}}{a} (ka+1)\, a_m^1 - 4\pi a^{\frac{3}{2}} f_{m+\frac{1}{2}}(ak)\frac{K_{m+\frac{1}{2}}(ak)}{\sqrt{a}} a'_m$$

$$- 4\pi A^{\frac{3}{2}}\sum_{n=2}^{n=\infty}\beta_n' f_{n+\frac{1}{2}}(Ak)\, A_{n,\,m,\,2}\frac{I_{m+\frac{1}{2}}(ak)}{\sqrt{a}} = 0.$$

For the purpose of obtaining an approximate solution, it will be sufficient to take account of β'_n when $n = 2$ only and hence, we obtain

$$a_m' = \frac{-A^{\frac{3}{2}}\beta'_2 f_{\frac{5}{2}}(Ak)\, A_{2;\,m,\,2}\, I_{\frac{5}{2}}(ak) / \sqrt{a}}{\left\{\left(a\cosh ak - \frac{\sinh ak}{k}\right)\frac{e^{-ak}}{k^2 a}(ka+1)+af_{m+\frac{1}{2}}(ak)\,K_{m+\frac{1}{2}}(ak)\right\}}$$

$$\tag{24}$$

where β_2' is given by (23).

By symmetry

$$\beta'_m = \frac{-a^{\frac{3}{2}}a_2' f_{\frac{5}{2}}(ak)\, A_{2,m,2}\, I_{\frac{5}{2}}(Ak) / \sqrt{A}}{\left\{\left(A\cosh Ak - \frac{\sinh Ak}{k}\right)\frac{e^{-Ak}}{k^2 A}(KA+1)+A f_{m+\frac{1}{2}}(Ak)\,K_{m+\frac{1}{2}}(Ak)\right\}}$$

$$\tag{25}$$

where a_2' is given by (22).

§ 11.

We have thus determined all the unknown constants a_n, β_n, a'_n, β'_n. Substituting the values of a'_n's, a'_n's in equation (12), we get the equation of the boundary of mass o. Similarly, substituting the values of β_n's, β_n'''s in equation (13), we get the equation of the boundary of the mass O.

Hence the equations of the two masses can be approximately written in the forms :

$$\frac{r}{a} = 1 + \frac{\dfrac{20\pi}{k^2\sqrt{c}}i^{-\frac{1}{2}}\left(A\cosh Ak - \dfrac{\sinh Ak}{k}\right)I_{\frac{5}{2}}(ak)K_{\frac{5}{2}}(kc) + \frac{1}{6}a^2\omega^2}{4\pi\left\{\dfrac{1}{k^2}\left(a\cosh ak - \dfrac{\sinh ah}{k}\right)\dfrac{e^{-ak}}{a}(ka+1) + a f_{\frac{5}{2}}(ak)\,K_{\frac{5}{2}}(ak)\right\}}\, p_2(\mu)$$

$$-\;\frac{\frac{1}{12}a^2\omega^2\, p_2^2(\mu)\cos 2\phi}{4\pi\left\{\dfrac{1}{k^2}\left(a\cosh ak - \dfrac{\sinh ak}{k}\right)\dfrac{e^{-ak}}{a}(ka+1) + a f_{\frac{5}{2}}(ak)\,K_{\frac{5}{2}}(ak)\right\}}$$

[Here we have neglected all terms involving $p_m^2(\mu)\cos 2\phi$ and $p_m(\mu)$ in which $m > 2$].

From symmetry,

$$\frac{R}{A} = 1 + \frac{\dfrac{20\pi}{k^2\sqrt{c}}i^{\frac{1}{2}}\left\{a\cosh ak - \dfrac{\sinh ak}{k}\right\}I_{\frac{5}{2}}(Ak)K_{\frac{5}{2}}(kc) + \frac{1}{6}A^2\omega^2}{4\pi\left\{\dfrac{1}{k^2}\left(A\cosh Ak - \dfrac{\sinh Ak}{k}\right)\dfrac{e^{-Ak}}{A}(kA+1) + A f_{\frac{5}{2}}(Ak)K_{\frac{5}{2}}(Ak)\right\}}\, P_2(\mu)$$

$$-\;\frac{\frac{1}{12}A^2\omega^2\, P_2^2(\mu)\cos 2\phi}{4\pi\left\{\dfrac{1}{k^2}\left(A\cosh Ak - \dfrac{\sinh Ak}{k}\right)\dfrac{e^{-Ak}}{A}(kA+1) + A f_{\frac{5}{2}}(Ak)K_{\frac{5}{2}}(Ak)\right\}}$$

On Aerial Waves generated by Impact

Part I*

By

Sudhansukumar Banerji

1. *Introduction*

Hertz, in his well-known paper† on the collision of elastic solids shows that when two bodies impinge on each other with moderate velocities, the elastic distortions are more or less entirely localized over the region of contact, and that the duration of impact, though in itself a very small quantity, is a large multiple of the gravest period of free vibrations of either body. It follows, therefore, that no appreciable vibrations of the solids are set up by the impact, and that all parts of the impinging bodies, except those infinitely close to the point of impact, move as parts of rigid bodies.

In a recent paper‡ Lord Rayleigh has investigated the circumstances of the first appearance of sensible vibrations in the case of two impinging spheres, and his results seem to show that if vibrations are excited at all, the leading term in the radial displacement at the point of contact during the early part of the collision is given by an expression of the type

$$\phi = \frac{3}{4}\frac{\sqrt{\pi}}{\cdot}\frac{k_2 a^{\frac{3}{2}}}{a_3 n^{\frac{7}{2}}}\cos\left(nt+\frac{\pi}{4}\right),$$

where $\dot{a}$ is the relative velocity of impact, a_3 is a certain constant which can be easily calculated from Lamb's theory, and

$$k_2 = \frac{4\sqrt{2r}.\,\mathrm{E}}{9}, \qquad n = \sqrt{\frac{\mathrm{E}}{\rho}}.\frac{\cdot 85\pi}{\sqrt{3}r},$$

r being the radius of the sphere, E the Young's modulus, and ρ the density.

The leading term due to the end of the collision is obtained from this by changing nt to $n(t-\tau)$, τ being the duration of impact.

Also the ratio of the maximum kinetic energy of vibrations to the energy before collision is approximately given by an expression of the type

$$R = \frac{1}{50} \cdot \frac{\dot{a}}{\sqrt{(E/\rho)}}.$$

Since $\sqrt{E/\rho}$ is the velocity of longitudinal vibrations along a bar of the material of the solids in question, we see that, in general, the expression for ϕ is very small in magnitude, and that R is an exceedingly small ratio.

Lord Rayleigh's results show that under ordinary conditions, that is, unless the spheres are very large in size or the relative velocity of impact is very great, vibrations should not be generated in appreciable degree, and that the energy of the colliding spheres remains translational. Moreover, even if vibrations be excited at all, the pitch of the gravest sound so produced would be very high, in fact almost beyond the range of audibility. For example, in the case of two mahogany balls of 6 cm. diameter, the frequency of the gravest vibrations excited would be about 37,000 per sec. We know, however, from experience that when two spheres, say two billiard-balls, impinge directly upon each other, aerial waves of considerable intensity are generated which are audible as the characteristic sound of impact. The investigation described in the present paper was undertaken to ascertain, both theoretically and experimentally, the origin and characteristics of the sound produced by such impact.

Since, as we have seen, under ordinary conditions vibrations cannot be excited in any perceptible degree, practically the whole of the sound of impact must be principally due to the impulse given to the fluid medium by the surfaces of the spheres, which undergo a sudden change of velocity as a result of the impact. The only alternative explanation that might be suggested is some kind of action, namely, a sudden compression or rarefaction in the neighbourhood of the region of contact; but this, it seems, can hardly be correct, as the spherical shape of the balls and the smallness of the relative velocity of impact would not readily admit of any specially intense compression or rarefaction being set up in the medium round the region of contact. Probably some kind of local reciprocating motion would be set up in this region, but this would not be of much importance.

The first hypothesis suggested in the preceding paragraph regarding the origin of the sound can be fully tested by an experimental and theoretical investigation of the distribution of intensities in different directions round the colliding spheres, and by studying the manner in which the sound depends (1) on the duration of the impact, (2) on the coefficient of restitution, (3) on the diameter of the balls, and (4) on the relative velocity of impact and possibly other factors also.

2. Measurement of the Intensity.

The distribution of intensities in different directions round the colliding spheres is found to possess many remarkable peculiarities which would be very difficult to reconcile with any other hypothesis regarding the origin of the sound. Even by the unaided ear one can perceive that the intensity of the sound is greatest when heard in the direction of movement of the colliding spheres, and is comparatively quite feeble in the plane at right angles to this line. Inside a laboratory the reflexions from the walls of the room give some trouble. The contrast between the intensities in the two directions is therefore best appreciated by the unaided ear when the observations are made in the open air, so as to avoid such reflexions as far as possible. A rough estimate of the ratio of the intensities can be made by varying the distance of the colliding spheres from the observer. So far as could be judged, the sound in the direction of impact appeared at least three or four times more intense in one direction than in the other. Some uncertainty was caused by the difference in the character of the sound from various directions, this difference being so marked that by its aid alone the angle made by the line of collision with the direction of the observer could be judged with fair accuracy. Other remarkable peculiarities were revealed when it was arranged to obtain a quantitative measurement of the relative intensities in actual experiment. It was then noticed that the intensity practically vanishes on a cone making an angle of about 67° with the line joining the centres.

After many trials an apparatus has been devised which appears to satisfy the necessary conditions of extreme sensitiveness, suitability for quantitative work, and convenience in actual use. This apparatus which is believed to be of a new type, is based on a ballistic principle. Its construction is quite distinct from that of the phonoscope invented by Dr. Erskine-Murray, or other similar devices in which the motion

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of a membrane or disk on which the sound-waves are incident deflects
a pivoted mirror connected with it. As a matter of fact, a phonos-
cope of the ordinary type was given the first trial, but proved quite
unsuitable for the present work, as the deflection observed with it
was too small and too sudden to be capable of measurement by
visual observation, or even for satisfactory photographic registra-
tion.

The apparatus finally devised and employed consists of a small
mirror attached to a pivoted axle whose free movement is controlled
by a fine spiral spring. (In practice the balance-wheel and hair-
spring of a watch proved very satisfactory, the mirror being attached
radially to the wheel with a little cement.) The sound is received
by a horn over the tubular end of which a mica disk is fixed. A
sharp metal pointer is fixed normally to the centre of the disk and
its end lightly touches the pivoted mirror referred to above, but is
not connected with it. The light from a slit illuminated by an
arc-lamp is condensed by a lens on the pivoted mirror, the reflected
light forming a sharp image of the slit on a distant graduated
screen. For the production of impact, the balls are hung side by
side by bifilar suspension from a frame work which is capable
of rotation round a vertical axis. The balls can be made to impinge
on each other in a direction making any desired angle with the
axis of the horn by simply rotating the framework. This angle can
be read off on a graduated circle fixed below the frame. In order
to obtain perfect regularity in the sound of the impact and to avoid
unnecessary reflexions from closely contiguous bodies, an electro-
magnetic arrangement was used by which the balls could be automati-
cally dropped on breaking the circuit.

As soon as the balls collide, the sound-wave generated by the
impact passes through the horn and impinges on the mica disk.
The motion of the pointer attached to the disk gives a kick to
the pivoted mirror, which moves off freely until it is brought to
rest by the controlling spiral spring. The mirror then comes back
to the pointer, which brings it to rest. The deflexion of the spot
of light on the distant screen gives a measure of the kick given to
the mirror. It is found that the apparatus is extremely sensitive,
very faint sounds being sufficient to produce deflexions which can
be read off by eye nearly as easily as those of a ballistic galvanometer.
Moreover, the behaviour of the mirror is very regular, and its motion

perfectly aperiodic. The mode of action of the apparatus described above can be verified by observing the motion of the mirror and the pointer under a low-power microscope.

One valuable feature of the apparatus is that it is quite unaffected by any echoes of the original sound of impact from the walls of the laboratory-room in which the experiments are made. This is because the pointer attached to the mica disk ceases to touch the mirror long before the echoes from the walls arrive at it. The results obtained by its use have been verified by working at different points within the room, and also in rooms of widely differing shape and size.

In order to fully understand the action of the mica disk and pointer, we have to study their forced vibrations under the influence of the sound-pulse. We shall presently come to this point. Meanwhile the results obtained by its use may be described.

Observations have been made of the deflexions shown by the apparatus when two balls impinge directly upon each other with

Fig. 1.

Inclination (in degrees) of the line of collision with axis of horn ⟶
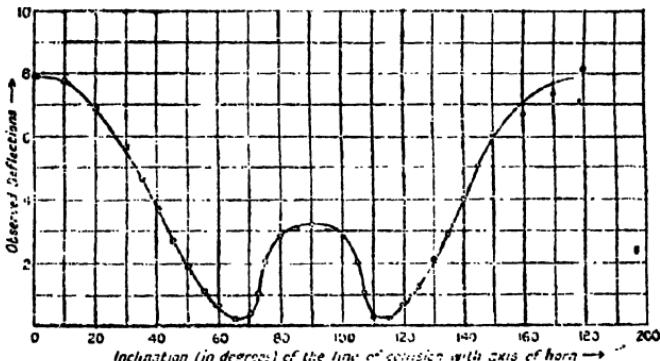
a given velocity in different directions with respect to the axis of the receiving horn. The results are recorded in fig. 1. The curve exhibits quite a number of remarkable peculiarities. It shows that the intensity is maximum in the line joining the centres, and that it gradully diminishes until it practically vanishes at an angle of about 67° with the line joining the centres, when it again increases rather abruptly until it attains a second maximum value at an angle of 90°. The experiment has been tried with pairs of spheres

of various materials, viz. (1) billiard balls, (2) marble, (3) aluminium, and (4) wood, and analogous results are noted in all the cases.

It is not difficult to see, in a general way, that the distribution of intensity shown in fig. 1 is in accordance with the hypothesis as to the origin of the sound with which we started. As a result of the impact the balls undergo very rapid changes of velocity in opposite directions. The case is somewhat analogous to the well-known effect of the zones of silence noticed in the neighbourhood of the prongs of a tuning-fork, but differs from it somewhat owing to the spherical shape of the balls, the non-periodic character of their motion, and their close contiguity at the instant of impact.

That the results shown in fig. 1 are quite reliable has been further tested by three methods. By measuring the deflexions of the spot of light for impacts at different distances from the mouth of the horn, the balls being made to impinge always with the same velocity and in the same direction, the deflexions are found practically to vary inversely as the square of the distances from the mouth of the horn.

Fig. 2.
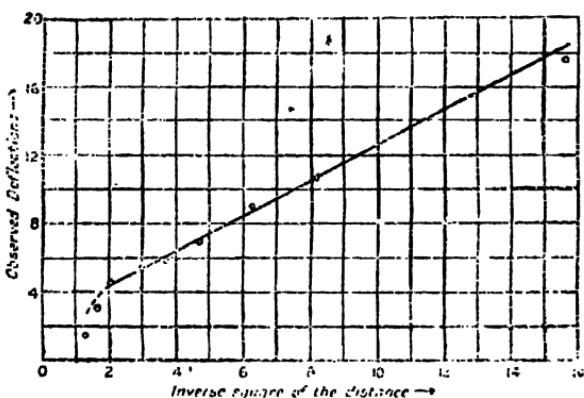
The results are shown in fig. 2, where squares of the reciprocal of the distances have been plotted against the deflexions. The curve shows that over a very wide range the deflexions, practically vary inversely as the square of the distances, and it is only when we come towards the origin that the curve shows a tendency to assume a parabolic shape. Further, measurement of the deflexions for different velocities of impact shows that within the range of the experiment the

deflexions vary directly as the squares of the velocities. The results
are shown in Table I. When the squares of the velocities are plotted

TABLE 1.

Nos.	Velocity of Impact.	Mean Deflexion
1.............................	23·06	8·63
2.............................	21·32	7·53
3..........	19·50	6·82
4.............	17·75	5·20
5........................ ..̇..........	10·80	1·43

against the deflexions they give practically a straight line passing
through the origin. Experiments have also been made with pairs of
balls of the same material but of different diameters. In this case it is
found that the deflexions vary practically as the fourth power of the
diameters of the balls. The results for the case of three pairs of
wooden balls are given in Table II. All these results show, as we
shall presently see, that the apparatus practically measures the
intensity of the sound produced by impact.

TABLE II.

Nos.	Diameters of the Balls.	Mean Deflexions.
	3 inches.	21·92
	2¼ inches..	7·02
	1½ inches.	1·35

3. Nature of the Wave-Motion.

A complete theoretical investigation of the nature of the wave-
motion started by impact, assuming that no vibrations are excited
in the balls, is beset with considerable mathematical difficulties. We
shall confine ourselves to the case of two equal balls. As a result of

the impact, the balls suffer changes of velocity U during the short interval of time known as the duration of impact, in consequence of which an impulsive pressure is communicated to the surrounding medium and a train of sound-waves is started travelling forward with a definite wave-front. As a simplification we shall assume that the change in velocity is instantaneously acquired by the balls. As a matter of fact, if we examine the curve for the relative velocities of the centres of mass for the period the balls are in actual contact, we notice that the most rapid changes in velocity occur only at the epoch of greatest compression, and as the duration of impact itself is a very small quantity (usually less than the 2000th part of a second), we see that the effect of the duration of the impact on the sound-waves is generally not of very great importance. At any rate, we are not wide of the mark in taking the change in velocity as practically instantaneous.

(1) The case of a single sphere.

We shall first consider the case when a single sphere suffers an instantaneous change in velocity U.

If M be the mass of the sphere, its equation of motion can be written in the form

$$M \frac{d^2 x}{dt^2} = - \iint p \cos \theta . a^2 \, d\omega, \quad \ldots \quad \ldots \quad (1)$$

where a is the radius of the sphere, p is the pressure at a point on the surface of the sphere, and $d\omega$ an elementary solid angle.

Also if ψ denote the velocity potential of the wave-motion started, the condition of continuity of normal motion on the surface of the sphere gives

$$\frac{\partial \psi}{\partial r} = - \frac{dx}{dt} \cos \theta, \text{ when } r = a \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The initial circumstances at time $t = 0$ give

$$x = 0 \text{ and } \frac{dx}{dt} = U. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Further, the condition of discontinuity at the spherical boundary of the advancing wave gives

$$\frac{\partial \psi}{\partial t} = - c \frac{\partial \psi}{\partial r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

to be satisfied for $r = ct + a$, c being the velocity of sound.

We can now assume for ψ the following expression

$$\psi = \frac{\partial}{\partial r} \cdot \frac{f(ct-r)}{r} \cdot \cos \theta, \quad . \quad . \quad (5)$$

and we can easily determine the arbitrary function involved in this expression by a method first given by Prof. Love[*] so as to satisfy all the conditions enumerated above.

The method consists in assuming

$$f(ct-r) = Ae^{\lambda(ct - r + a)} \text{ and } x = Be^{\lambda ct},$$

and then on substitution in the boundary conditions (1) and (2), we notice that λ satisfies a biquadratic equation, two of whose roots are zero. The constants A's and B's are then determined with the help of the remaining conditions.

If we assume that the ratio of the mass of the air displaced by the sphere to its own mass is a very small quantity, we see that the expression for ψ can be written in the simple form

$$\psi = A \cdot \left(\frac{1}{r}\right)\cos\theta - \frac{\sqrt{2}\,Ua^3}{4} \frac{\partial}{\partial r}\left[\frac{e^{-\left(\frac{ct+a-r}{a}\right)}}{r}\cos\left(\frac{ct+a-r}{a}-\frac{1}{4}\pi\right)\right]\cos\theta$$

$$. \quad . \quad . \quad . \quad (6)$$

where A is an indeterminate constant.

The first term in this expression is a degenerated function which does not satisfy the usual differential equation for wave-propagation, and consequently does not represent a wave-disturbance. This term arises form the subsequent motion of the sphere with a nearly constant velocity which involves only a local reciprocating motion of the neighbouring air.

The wave-motion produced is therefore given by the expression

$$\psi = -\sqrt{2}U\,a^3 \cdot \frac{\partial}{\partial r}\left[e^{\left(\frac{ct+a-r}{a}\right)}\cos\left(\frac{ct+a-r}{a}-\frac{1}{4}\pi\right)\right]\cos\theta. \quad . \quad . \quad (7)$$

Thus we see that the wave-motion generated by an instantaneous change in velocity of a single sphere is of the damped harmonic type which is practically confined to a small region near the front of the advancing wave.

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[*] Love, " Some Illustrations of the Modes of Decay of Vibratory Motions," Proc. Lond. Math. Soc. (2) vol. ii. p. 88 (1904). [See also Lamb's 'Hydrodynamics,' Art. 295.]

(2) The case of two spheres.

The solution for the case when both the spheres undergo instantaneous change in velocity U cannot, however, be so easily obtained. In a recent paper[*] published in the 'Bulletin' of the Calcutta Mathematical Society, I have given a method by which a solution for this case can be obtained. For our present purpose, however, we see from symmetry that if we take as our origin the point of contact of two equal impinging spheres, the velocity potential of the wave-motion which satisfies the boundary conditions over the surfaces of both the spheres can be written in the form

$$\psi = A_0 \frac{U a^2}{r} e^{\lambda_0 \left(\frac{ct-r}{a}\right)} + A_2 U a^4 \left(\frac{\partial}{r \partial r}\right)^2 \left\{ \frac{e^{\lambda_2 \left(\frac{ct-r}{a}\right)}}{r} \right\} \cdot r^2 P_2 (\cos \theta)$$

$$+ \&c. + A_{2n} U a^{2n+2} \left(\frac{\partial}{r \partial r}\right)^{2n} \left\{ \frac{e^{\lambda_{2n} \left(\frac{ct-r}{a}\right)}}{r} \right\} \cdot r^{2n} P_{2n} (\cos \theta)$$

$$+ \&c., \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

due regard being paid to the dimensions of both the sides. A_0, A_2, &c.; λ_0, λ_2, &c., are certain constants not depending on the radius of the spheres to be determined by the boundary conditions.

At a great distance from the source of sound we can neglect all powers of $\frac{1}{r}$ in this expression beyond the first. So that at a great distance we have approximately

$$\psi = \frac{U a^2}{r} \left[A_0 e^{\lambda_0 \left(\frac{ct-r}{a}\right)} + A_2 \lambda_2^2 e^{\lambda_2 \left(\frac{ct-r}{a}\right)} P_2 (\cos \theta) + \&c. \right] \quad . \quad . \quad (9)$$

Also, since the sound-pulse is practically confined to the wave-front, and also since in this region we have either r equal to ct or less than ct by a few diameters, we see that the expression within the bracket may be regarded as a simple function of the time and the inclination θ, independent of the radius of the spheres or the distance r.

The intensity of the sound for large values of r therefore varies as $\dfrac{U^2 a^4}{r^2}$.

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* On " Sound-waves due to prescribed Vibrations on a spherical Surface in the presence of a rigid and fixed Spherical Obstacle," Bulletin of the Calcutta Mathematical Society, vol. iv.

We thus arrive at the following results :—

(1) The intensity of the sound varies as the square of the change in velocity of the colliding spheres.

(2) It varies inversely as the square of the distance from the point of contact of the spheres.

(3) It varies as the fourth power of the radius of the spheres.

The truth of these results has already been verified experimentally, provided we assume that the apparatus measures the intensity, which we shall presently see it does.

We can easily study the forced and the free vibrations of the mica disk under the action of the sound-pulse. The forced and free vibrations of a membrane and those of a telephone plate have been studied by various writers. Without entering into mathematical details, we see that the disturbance produced by impact travels forward as a sound-pulse of the damped harmonic type which is sensible only within a few diameters from the inner side of the boundary of the advancing wave. Its action on the mica disk, which has usually a smaller natural frequency of vibration than that of the waves, is so very sudden and lasts for so short a time that the whole effect partakes of the character of an impulsive pressure in consequence of which the mica disk suddenly acquires a velocity and free vibrations of considerable amplitude are excited in it. Assuming that the mica disk is not displaced considerably, we can easily see from elementary considerations that since $\rho \dfrac{\partial \psi}{\partial t}$ is the pressure per unit area on the mica disk, were ψ is the velocity potential of the sound-pulse and ρ the density of air, the initial velocity communicated to the mica disk would be practicallay proportional to the quantity

$$\int_{t_0} \rho \frac{\partial \psi}{\partial t}\, dt,$$

t_0 being the instant when the sound-pulse meets the mica disk. When this velocity attains the maximum value, the mirror leaves the pointer and moves with that maximum velocity. This velocity is therefore proportional to the quantity $[\rho\psi]_{t_0}^{t}$, the instant t being so chosen that this expression has the maximum value. If we denote this quantity by v and the deflexion of the mirror by θ, then we

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must have v and θ connected by the relation

$$v^2 = a\theta + b\theta^2,$$

where a and b are two constants depending on the elasticity of the spring and its initial strained condition. In actual practice the mirror is initially in contact with the pointer with a sensible pressure, and as the deflexion is usually very small, the second term in the above expression is nearly negligible in comparison with the first. v^2 is, therefore, practically proportional to $a\theta$; in other words, the angular deflexion of the mirror is approximately proportional to the intensity of the sound incident on it.

On account of mathematical difficulties, it seems to be a hopeless task to attempt a numerical calculation of the distribution of intensities in different directions round the colliding spheres. But the analogous problem of two vibrating spheres whose distance apart varies periodically presents features similar to this problem, when the wave-length of the disturbance produced is sufficiently small. For this case, however, we can approximately calculate the distribution of intensities in different directions by the following method.

If we take as our origin the point symmetrically situated between the two spheres and the line joining the centres as our initial line, it is easy to see that the velocity potential of the wave-motion will be given by

$$\psi = \left[A_0 f_0(kr) + A_2 \frac{(kr)^2 f_2(kr)}{\frac{d}{da}\{(ka)^2 f_2(ka)\}} P_2(\cos\theta) \right.$$

$$\left. + A_4 \frac{(kr)^4 f_4(kr)}{\frac{d}{da}\{(ka)^4 f_4(ka)\}} P_4(\cos\theta) + \&c. \right] e^{ikct},$$

where

$$f_n(\zeta) = \frac{i^n e^{-i\zeta}}{\zeta^{n+1}} \left\{ 1 + \frac{n(n+1)}{2i\zeta} + \frac{(n-1)n(n+1)(n+2)}{2\,.\,4\,.\,(i\zeta)^2} \right.$$

$$\left. + \ldots + \frac{1\,.\,2\,.\,3\ldots 2n}{2\,.\,4\,.\,6\ldots 2n(i\zeta)^n} \right.$$

and $\frac{2\pi}{k}$ is the wave-length.

The unknown constants A_0, A_2, &c. have to be determined by means of the condition of continuity of normal motion on the surfaces

of the spheres. We see from this expression that the disturbance produced at any point due to this system of two vibrating spheres will be the same as that due to a prescribed vibration given by

$$[A_0 + A_2 P_2(\cos\theta) + A_4 P_4(\cos\theta) + \dots]e^{ik\,ct},$$

on the surface of a single sphere of radius equal to that of either sphere and having its centre at the origin. Now, from a consideration of the nature of the motion produced in the immediate neighbourhood of the two spheres, we can easily ascribe approximate values to the constants A_0, A_2, &c., which will conform as nearly as possible to the true state of affairs. As a first approximation, we can represent this disturbance by $U\cos 2\theta \, .e^{ik\,rt}$, which is the same as

$$U\,[\tfrac{4}{3}P_2(\cos\theta) - \tfrac{1}{3}P_0(\cos\theta)]e^{ik\,ct},$$

so that $A_0 = -\tfrac{1}{3}$, $A_2 = \tfrac{4}{3}$ and the rest vanishes. This type of vibration shows that while the two caps bounded by the parallels of latitude of 45° and 135° are moving outwards the intermediate zone is moving inwards and *vice versa*.

Now, if we assume various values for the quantity ka which will determine the wave-length for a particular pair of balls, we can easily calculate the values of the intensity at a great distance from the source of sound. First suppose that $ka = 1$, then since at a great distance

$$f_n(kr) = \frac{i}{(kr)^n},$$

we have

$$\psi = -\left[\frac{A_0}{2}(1-i) - \frac{A_2}{89}(5-8i)\,P_2(\cos\theta)\right.$$

$$\left. + \frac{A_4}{402337}(296-561i)P_4(\cos\theta) + \&c. \right]\frac{e^{ik(ct-r-a)}}{r}$$

Denoting the real and the imaginary parts of the expression within the bracket by F and G, we have

$$F = \tfrac{1}{2}A_0 - \frac{5A_2}{89}P_2(\cos\theta) + \frac{296A_4}{402337}P_4(\cos\theta) + \&c.,$$

$$G = -\tfrac{1}{2}A_0 + \frac{8A_2}{89}P_2(\cos\theta) - \frac{561A_4}{402337}P_4(\cos\theta) + \&c.$$

Since the intensity is proportional to $F^2 + G^2$, we see from the above expressions on substituting the values of A_0, A_2, &c., that the intensity is maximum in the direction of the line joining the centres, and that it gradually decreases and assumes the minimum value in the perpendicular direction.

Now, if we further diminish the wave-length—that is, if we assume $ka = 2$, we get

$$\psi = 100 \left[A_0 \frac{25-50i}{3125} + A_2 \frac{44+62\cdot5i}{5842\cdot25} P_2(\cos\theta) \right.$$

$$\left. + A_4 \frac{405+1170\cdot6i}{1534329\cdot36} P_4(\cos\theta) + \&c. \right] \frac{e^{ik\ (ct-r+a)}}{r},$$

$$= \frac{1}{30} \left[3A_0(8-16i) + 3A_2(7\cdot53+10\cdot7i)P_2(\cos\theta) \right.$$

$$\left. + \&c. \right] \frac{e^{ik\ (ct-r+a)}}{r}.$$

Hence

$$F = -8 + 30\cdot12 P_2(\cos\theta) + \&c.,$$

$$G = 16 + 42\cdot8 P_2(\cos\theta) + \&c.$$

We thus see that in this case $F^2 + G^2$ is maximum at $0°$ where its value is $3941\cdot44$ nearly, and that its value gradually decreases and assumes the minimum value nearly 116 at an angle of $61°$, and that it again increases and assumes a second maximum value at an angle of $90°$, where its value is $558\cdot16$ nearly. If we further decrease the wave-length we get results analogous to the above case, namely, that the intensity is maximum in the line joining the centres, and that it assumes a minimum value at some angle intermediate between $0°$ and $90°$ and a second maximum value at $90°$ which is much less than the first maximum. We can easily proceed to a second approximation by determining the values of the coefficients in the expression for the prescribed vibration on the surface of the imaginary sphere so as to agree as closely as possible with the actual state of affairs.

4.　Experimental study of the character of the sound-wave.

The experimental results described before would be very difficult to explain on any hypothesis other than that which we have assumed

for the origin of the sound.   This hypothesis may therefore be regarded as confirmed by experiment to the practical exclusion of any others.   It was, however, considered that further experimental study of the character of the sound-wave emitted in different directions by the colliding spheres would be of very great interest.

Attempts have been made to obtain photographic records of the motion of the mica disk under the action of the sound-wave by various methods.   The first, due to Siegbahn*, was that of optically recording the motion of a pointer attached to the disk by the use of two microscopes focussed on it, one on either side.   Another method which was used was simply to fix a small mirror to the mica disk at the place of greatest angular deflexions and to photograph on a falling plate the motions of the image of an illuminated slit formed by reflexions from the surface of the mirror.   While both of these methods gave results confirming the broad indications of theory, the photographs obtained could not be regarded as satisfactory records of the character of the sound-wave owing to the free vibrations of the mica disk excited by the sound of the impact which continued for an appreciable period.   Even the first two or three swings, which were much larger in amplitude than the others, showed the free vibrations somewhat prominently. ⁻ This was evidently due to the highly implusive character of the sound-wave.   Some improvement was obtained by using a disk stiffened by attachment to a wire stretched in front of it under tension.   (This was taken out from an old gramophone.)   The motion of the wire was recorded photographically.   Even with this arrangement, however, the free vibrations of the receiving apparatus were prominent.   Unless a sufficiently sensitive and at the same time strongly damped recorder is found, there does not appear much hope of obtaining a satisfactory direct record of the character of the sound-pulse.   The writer hopes shortly to try the use of an acetylene-gas manometric flame and the phonodeik invented by Dr. Mitter and will also make further experiments with receivers of various types for obtaining an accurate record of the character of the sound-wave.

Experiments have also been made with the ballistic apparatus described in the first part of the paper to compare the effects of the impacts of balls of the same size but of different materials. On trying the effect of a pair of wooden balls, and

* Phil. Mag., May, 1914.

those of a pair of billiard balls of equal size, the latter were found to give a deflexion about twice the deflexion due to the former. In this we may trace the effect not only of the larger coefficient of restitution of the billiard balls, but also probably of the shorter duration of impact which would be more effective in setting up an impulsive wave-motion in the fluid.

5. *Summary and Conclusion.*

The intensity of the sound generated by the collision of two solid spheres varies very greatly in different directions relative to the line of impact and the character of the sound shows a similar pronounced variation. This observation was first made with the unaided ear and communicated to me by Prof. C. V. Raman, and the present work was undertaken at his suggestion to investigate this effect in detail, both theoretically and experimentally. A new type of apparatus in which the ballistic principle is utilized has been used to investigate the intensity of the sound in different directions. The results show a maximum intensity in the line of collision, pactically zero intensity on the surface of the cone of semi-vertical angle 67°, and a second, but feeble maximum in the plane perpendicular to the line of impact. These results combined with the indications of theory and further observations on the character of the sound-wave show that, practically speaking, it is produced entirely by the accelerated motion of the spheres during the impact. The law of variation of the intensity of the sound with the velocity of impact and the radius of the balls has also been found and tested exprimentally. The writer hopes later on to carry out further work on the subject, particularly in the matter of getting direct records of the character of the sound-wave and observing the effects of oblique impact and the impact of spheres of unequal diameters.

CALCUTTA,
28th January, 1916.

On Aerial Waves Generated by Impact.*

Part II.

By

Sudhansukumar Banerji.

1. *Introduction.*

The origin and characteristics of the sound produced by the collision of two solid spheres were discussed by me at some length in the first paper under the same title that was published in the Philosophical Magazine for July, 1916. It was shown in that paper that the sound is not due to the vibrations set up in the spheres, which, in any ordinary material, are both too high in pitch to be audible, and too faint in intensity, but to aerial waves set up by the reversal of the motion of the spheres as a whole. The intensity of the sound in different directions for the case, in which the two spheres were of the same material and diameter, was investigated by the aid of a new instrument which will be referred to as " the Ballistic Phonometer."† The intensity was found to be a maximum along the line of collision, falling off gradually in other directions to a value which is practically zero on the surface of a cone of semi-vertical angle 67° and rising again to a second but feebler maximum in a plane at right angles to the line of collision.

In view of the interesting results obtained for the case of two equal spheres, it was arranged to continue the investigation and to measure the distribution of intensity when the colliding spheres were not both of the same radius or material. A mathematical investigation of the nature of the results to be expected in these cases was also undertaken. In order to exhibit the results of the experiment and of the theoretical calculation, a plan has now been adopted which is much more suitable than the one used in the first paper. This will be best understood by reference to fig. 1, which refers to the case of two spheres of the same material and diameter.

Fig. 1.
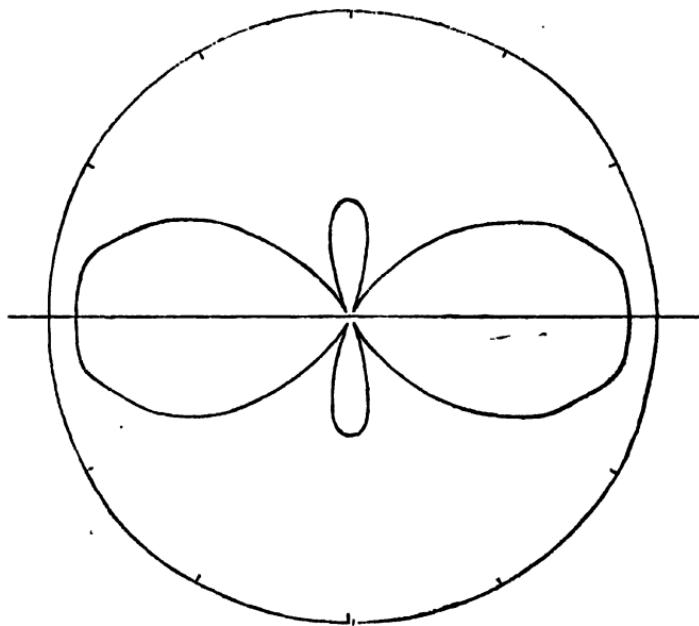
Observed distribution of sound intensity around two equal
colliding spheres of the same material.

This figure has been drawn by taking the point at which the spheres impinge as origin and the line of collision as the axis of x, and setting off the indications of the Ballistic Phonometer as radii vectores at the respective angles which the direction in which the sound is measured makes with the line of collision. The curve thus represents the distribution of intensity round the colliding spheres in polar co-ordinates, the points at which the intensity of the sound is measured being assumed to be all at the same distance from the spheres. The results are brought much more vividly before the eye by a diagram of this kind than by plotting the results on squared paper.

2. Case of two spheres of the same material but of different diameters.

Figure 2, which shows the observed distribution of intensity when two spheres of wood of diameters 3 inches and 2¼ inches collide with each other, is typical of the results obtained when the impinging spheres are nearly of the same density and are of different diameters. There is distinct asymmetry about a plane perpendicular to the line of impact. In addition to the maxima of intensity in the two directions of the line of collision, we have the maxima in lateral directions, which are not at right angles to this line. The directions in which the intensity is a minimum are also asymmetrically situated.

For the explanation of these and other results, we have naturally to turn to the mathematical theory, which rests upon the fact that the sound is due to the wave-motion set up in the fluid by the sudden reversal of the motion of the spheres. Let a and b be the radii of the two spheres and ρ_a and ρ_b be their densities. Then the masses of the spheres are $\frac{4}{3}\pi\rho_a a^3$ and $\frac{4}{3}\pi\rho_b b^3$ respectively. Denoting the changes in velocity which the spheres undergo as a result of the impact by U_a and U_b respectively, by the principle of constant momentum we have $U_a/U_b = \rho_b b^3/\rho_a a^3$. The ratio $\frac{U_a}{U_b}$ thus depends only on the diameters and the densities of the spheres, while, of course, the actual values of U_a and U_b would depend on the relative velocity before impact and the co-efficient of restitution. It is obvious that if we leave out of account the duration of impact, that is, regard the changes in velocity of the spheres as taking place practically instantaneously, the character and the ratio of the

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Fig. 2.
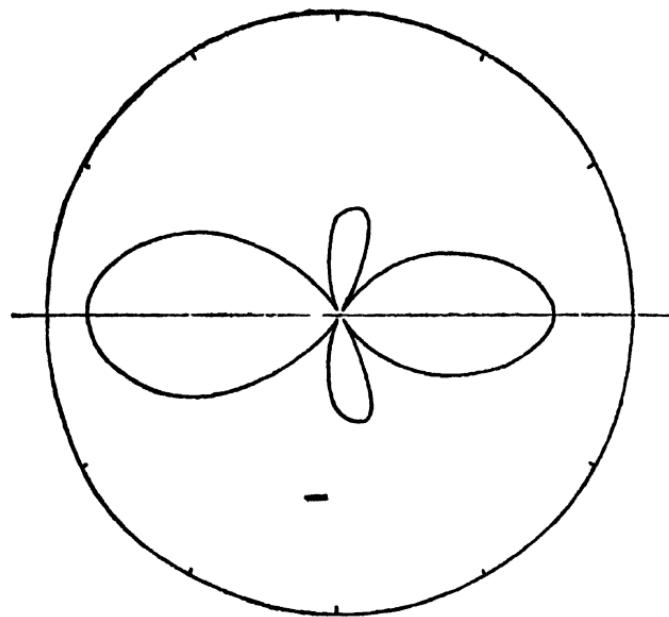
Observed distribution of intensity due to impact of two unequal
spheres of wood.

Sphere on left : 3 inches diameter.

Sphere on right : 2¼ inches diameter.

intensities of the sound produced in different directions would be completely determined by the sizes of the spheres and the ratio of their changes of velocity, that is, by *their diameters* and *their masses ;* when the spheres are of the same material, the nature of the motion in the fluid set up by the impact depends only on the radii of the

The mathematical problem of finding the nature of the fluid motion set up by the reversal of the motion of the spheres, taking the finite duration of impact into account, would appear to be of great difficulty. In my first paper, I have shown that when a single sphere of radius a undergoes an instantaneous change of velocity U, the wave motion produced is given by the expression

$$\psi = -\frac{\sqrt{2}Ua^3}{4} \frac{\partial}{\partial r}\left[\frac{e^{-\left(\frac{ct+a-r}{c}\right)}}{r} \cos\left(\frac{ct+a-r}{a} - \frac{1}{4}\pi \right) \right] \cos\theta, \quad (1)$$

which indicates that it is of the damped harmonic type confined to a small region near the front of the advancing wave. The wave motion, set up in the case of two spheres in contact assumed to undergo instantaneous changes of velocity, would be of a more complicated type. In order to obtain a general idea of the nature of the results to be expected, particularly as to the intensity and character of the sound in different directions, we may consider the analogous acoustical problem of two rigid spheres nearly in contact, which execute small oscillations to and fro on the line of their centres. This problem may be mathematically formulated and approximately solved in the following manner :—

Given prescribed vibrations

$$U_a \cos\theta_1 e^{ikct} \text{ and } U_b \cos\theta_2 e^{ikct}$$

on the surfaces of two spheres of radii a and b, nearly in contact, it is required to determine the velocity potential of the wave motion started and the distribution of intensities round the spheres, where θ_1 and θ_2 are the angles measured at the centres A and B of the two spheres in opposite senses from the line joining the centres of the spheres.

Supposing now that an imaginary sphere is constructed, which is of just sufficient radius to envelop the two actual spheres (touching them externally), it is possible from a consideration of the nature of the motion that takes place in the immediate neighbourhood of the two spheres, to determine the vibrations on the surface of this imaginary sphere, which would produce on the external atmosphere

the same effect as the vibrations on the surfaces of the real spheres A and B. When the equivalent vibration on the surface of the enveloping sphere has been obtained, we can, by the use of the well-known solution for a single sphere, at once determine the wave motion at any external point.

The radius of the enveloping sphere is evidently $a + b$, and its centre is at a point C, such that $BC = a$ and $CA = b$.

If the point C be taken as origin, and if the equivalent vibration on the surface of the enveloping sphere be expressed by the series

$$\Sigma\ A_n\ P_n\ (\cos\theta)\ e^{ikct}, \tag{2}$$

where A_n's are known constants, the velocity potential of the wave motion produced at any external point is given by

$$\psi = -\frac{(a+b)^2}{r}\ e^{ik(ct-r+a+b)} \Sigma\ \frac{A_n P_n(\cos\theta)}{F_n(ik.\overline{a+b})}\ f_n\ (ikr), \tag{3}$$

where

$$f_n\ (ikr) = 1 + \frac{n\ (n+1)}{2.\ ikr} + \frac{(n-1)\ n\ (n+1)\ (n+2)}{2.\ 4.\ (ikr)^2} + \ldots$$

$$+ \frac{1.2.3.\ldots 2n}{2.4.6\ldots 2n.\ (ikr)^n},$$

and

$$F_n\ (ikr) = (1 + ikr)\ f_n\ (ikr) - ikr\ f_n'\ (ikr). \tag{4}$$

To obtain the equivalent vibrations on the surface of the enveloping sphere, we shall regard the small quantity of fluid enclosed by this sphere as practically *incompressible*, and use the well-known solution by the method of successive images for two spheres in an incompressible fluid.

We know that the velocity potential due to such a system of two spheres in an incompressible fluid can be expressed in the form

$$U_a\ \phi + U_b\ \phi', \tag{5}$$

where ϕ and ϕ' are to be determined by the conditions

$$\nabla^2\phi = 0,\ \nabla^2\ \phi' = 0,$$

$$\frac{\partial\phi}{\partial r_1} = -\cos\theta_1,\ \text{and}\ \frac{\partial\phi'}{\partial r_1} = 0\ \text{,when}\ r_1 = a,$$

$$\frac{\partial\phi'}{\partial r_2} = -\cos\theta_2,\ \text{and}\ \frac{\partial\phi}{\partial r_2} = 0,\ \text{when}\ r_2 = b,$$

r_1 and r_2 being radii vectores measured from A and B.

When ϕ and ϕ' have been determined so as to satisfy these conditions the equivalent vibrations on the surface of the imaginary enveloping sphere can be taken to be very approximately given by

$$-\left[U_a \frac{\partial \phi}{\partial r} + U_b \frac{\partial \phi'}{\partial r} \right]_{r=a+b}^{ikct} \quad \text{(6)}$$

The functions ϕ and ϕ' as is well-known can be determined by the method of successive images and if the expressions for the velocity potential due to these images be all transferred to the co-ordinates (r, θ) referred to the centre C of the imaginary enveloping sphere, we easily obtain

$$2\phi = a^3 \left[1 - \frac{b^3}{(a+b)^3} + \frac{b^3}{(a+2b)^3} - \frac{b^3}{(2a+2b)^3} + \frac{b^3}{(2a+3b)^3} - \dots \right]$$

$$\frac{P_1(\cos\theta)}{r^2}$$

$$+2a^3 \left[b - \frac{b^3(a^2+ab-b^2)}{(a+b)^4} + \frac{b^3(2b^2-a^2)}{(a+2b)^4} - \frac{b^3(2a^2+ab-2b^2)}{(2a+2b)^4} \right.$$

$$\left. + \frac{b^3(3b^2-2a^2)}{(2a+3b)^4} - \dots \right] \frac{P_2(\cos\theta)}{r^3}$$

$$+3a^3 \left[b^2 - \frac{b^3(a^2+ab-b^2)^2}{(a+b)^5} + \frac{b^3(2b^2-a^2)^2}{(a+2b)^5} - \frac{b^3(2a^2+ab-2b^2)^2}{(2a+2b)^5} \right.$$

$$\left. + \frac{b^3(3b^2-2a^2)^2}{(2a+3b)^5} - \dots \right] \frac{P_3(\cos\theta)}{r^4}$$

$$+4a^3 \left[b^3 - \frac{b^3(a^2+ab-b^2)^3}{(a+b)^6} + \frac{b^3(2b^2-a^2)^3}{(a+2b)^6} - \frac{b^3(2a^2+ab-2b^2)^3}{(2a+2b)^6} \right.$$

$$\left. + \frac{b^3(3b^2-2a^2)^3}{(2a+3b)^6} - \dots \right] \frac{P_4(\cos\theta)}{r^5}$$

$$+ \text{etc.,} \quad \text{(7)}$$

$$2\phi' = -b^3 \left[1 - \frac{a^3}{(b+a)^3} + \frac{a^3}{(b+2a)^3} - \frac{a^3}{(2b+2a)^3} + \frac{a^3}{(2b+3a)^3} - \dots \right]$$

$$\frac{P_1(\cos\theta)}{r^2}$$

$$+2b^3 \left[a - \frac{a^3(b^2+ab-a^2)}{(b+a)^4} + \frac{a^3(2a^2-b^2)}{(b+2a)^4} - \frac{a^3(2b^2+ab-2a^2)}{(2b+2a)^4} \right.$$

$$\left. + \frac{a^3(3a^2-2b^2)}{(2b+3a)^4} - \dots \right] \frac{P_2(\cos\theta)}{r^3}$$

Fig. 3.
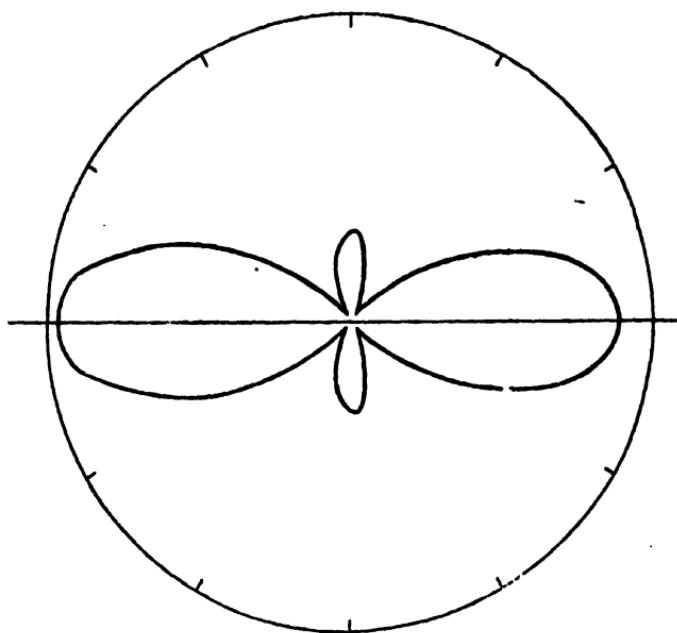
Calculated Form of Intensity Curve due to two Spheres of
diameters $2:1$. $[k(a+b)=2.]$

$$-3b^3\left[a^3-\frac{a^3(b^3+ab-a^3)^3}{(b+a)^5}+\frac{a^3(2a^3-b^3)^3}{(b+2a)^5}-\frac{a^3(2b^3+ab-2a^3)^3}{(2b+2a)^5}\right.$$
$$\left.+\frac{a^3(3a^3-2b^3)^3}{(2b+3a)^5}-\ldots\right]\frac{P_3\ (\cos\theta)}{r^4}$$

$$+4b^3\left[a^3-\frac{a^3(b^3+ab-a^3)^3}{(b+a)^6}+\frac{a^3(2a^3-b^3)^3}{(b+2a)^6}-\frac{a^3(2b^3+ab-2a^3)^3}{(2b+2a)^6}\right.$$
$$\left.+\frac{a^3(3a^3-2b^3)^3}{(2b+3a)^6}-\ldots\right]\frac{P_4\ (\cos\theta)}{r^5}$$

—etc.,
$$(8)$$

the law of formation of the series within the brackets being obvious.

Coming now to the present problem of two unequal spheres of the same material, let us take

$$a=2 \text{ inches and } b=1 \text{ inch.}$$

Since the changes of velocities of the two balls are inversely proportional to their masses, we must have

$$U_b=8U_a.$$

Substituting the values for a and b we easily find that

$$2\phi=2^3\left[\left(1+\frac{1}{4^3}+\frac{1}{7^3}+\frac{1}{10^3}+\ldots\right)-\left(\frac{1}{3^3}+\frac{1}{6^3}+\frac{1}{9^3}+\ldots\right)\right]\frac{P_1\ (\cos\theta)}{r^2}$$

$$+2\cdot2^3\left[1-\left(\frac{5}{3^4}+\frac{8}{6^4}+\frac{11}{9^4}+\frac{14}{12^4}+\ldots\right)-\left(\frac{2}{4^4}+\frac{5}{7^4}+\frac{8}{10^4}+\ldots\right)\right]\frac{P_2\ \cos\theta}{r^3}$$

$$+3\cdot2^3\left[1+\frac{2^3}{4^5}+\frac{5^3}{7^5}+\frac{8^3}{10^5}+\ldots\right)-\left(\frac{5^3}{3^5}+\frac{8^3}{6^5}+\frac{11^3}{9^5}+\ldots\right)\right]\frac{P_3\ (\cos\theta)}{r^4}$$

$$+4\cdot2^3\left[1-\left(\frac{5^3}{3^6}+\frac{8^3}{6^6}+\frac{11^3}{9^6}+\frac{14^3}{12^6}\right)+\ldots\right)-\left(\frac{2^3}{4^6}+\frac{5^3}{7^6}+\frac{8^3}{10^6}+\ldots\right)\right]$$

$$\frac{P_4\ (\cos\theta)}{r^5}$$

+etc.,
$$(9)$$

$$2\phi'=-2^3\left[\left(\frac{1}{2^3}+\frac{1}{5^3}+\frac{1}{8^3}+\frac{1}{11^3}+\ldots\right)-\left(\frac{1}{3^3}+\frac{1}{6^3}+\frac{1}{9^3}+\ldots\right)\right]\frac{P_1\ (\cos\theta)}{r^2}$$

$$+2\cdot2^3\left[\left(\frac{1}{3^4}+\frac{4}{6^4}+\frac{7}{9^4}+\ldots\right)+\left(\frac{4}{2^4}+\frac{7}{5^4}+\frac{10}{8^4}+\frac{13}{11^4}+\ldots\right)\right]\frac{P_2\ (\cos\theta)}{r^3}$$

$$-3\cdot2^3\left[\left(\frac{4^3}{2^5}+\frac{7^3}{5^5}+\frac{10^3}{8^5}+\frac{13^3}{11^5}+\ldots\right)-\left(\frac{1^3}{3^5}+\frac{4^3}{6^5}+\frac{7^3}{9^5}+\ldots\right)\right]\frac{P_3(\cos\theta)}{r^4}$$

$$+4\cdot2^3\left[\left(\frac{1^3}{3^6}+\frac{4^3}{6^6}+\frac{7^3}{9^6}+\ldots\right)+\left(\frac{4^3}{2^6}+\frac{7^3}{5^6}+\frac{10^3}{8^6}+\frac{13^3}{11^6}+\ldots\right)\right]\frac{P_4(\cos\theta)}{r^5}$$
—etc.
$$(10)$$

Summing the series, we easily find that the prescribed vibration on the surface of the imaginary enveloping sphere

$$U_s \left[\frac{\partial \phi}{\partial r} + 8 \frac{\partial \phi'}{\partial r} \right]_{r=3 \text{ ins.}} e^{ikct}$$

is proportional to

$$[\cdot 496 \, P_1 \, (\cos \theta) + 3\cdot 180 \, P_2 \, (\cos \theta) - 1\cdot 708 \, P_3 \, (\cos \theta) + 2\cdot 600 \, P_4 \, (\cos \theta) + \text{etc.}] e^{ikct}$$

We have seen that when the prescribed vibration on the surface of the imaginary sphere is

$$\Sigma \, A_n \, P_n \, (\cos \theta). e^{ikct},$$

the velocity potential of the wave-motion is

$$\psi = - \frac{(a+b)^2}{r} e^{ik(ct - r + a + b)} \Sigma \frac{A_n P_n (\cos \theta)}{F_n (ik. \overline{a+b})} f_n \, (ikr).$$

Now when r is large,

$$f_n \, (ikr) = 1,$$

so that the factor on which the relative intensities in various directions depend is

$$\Sigma \, A_n \frac{P_n (\cos \theta)}{F_n (ik. \overline{a+b})}.$$

Thus if we put this quantity $= F + i \, G$, the intensity of the vibrations in various directions is measured by $F^2 + G^2$.

The distribution of intensities in different directions round the spheres will be influenced to a considerable extent by the value of the wave-length chosen. If we take $k \, (a+b) = 2$, the wave-length is 3π inches and if we take $k \, (a+b) = 3$, the wave-length is 2π inches. From the expression (1) for the wave-motion produced by a single sphere undergoing an instantaneous change of velocity, it is seen that the wave-length to be chosen is of the same order as the circumference of the sphere. From this, it appears that for a system of two spheres whose radii are 1 inch and 2 inches respectively, the wave-length to be chosen should be some value intermediate between 2π and 4π, probably nearer 2π than 4π; for, in the actual case of impact, the smaller ball which would undergo by far the greater change in velocity would probably influence the character of the motion to a greater extent than the larger sphere. At the same time it must not be forgotten that the analogy between the case

Fig. 4.
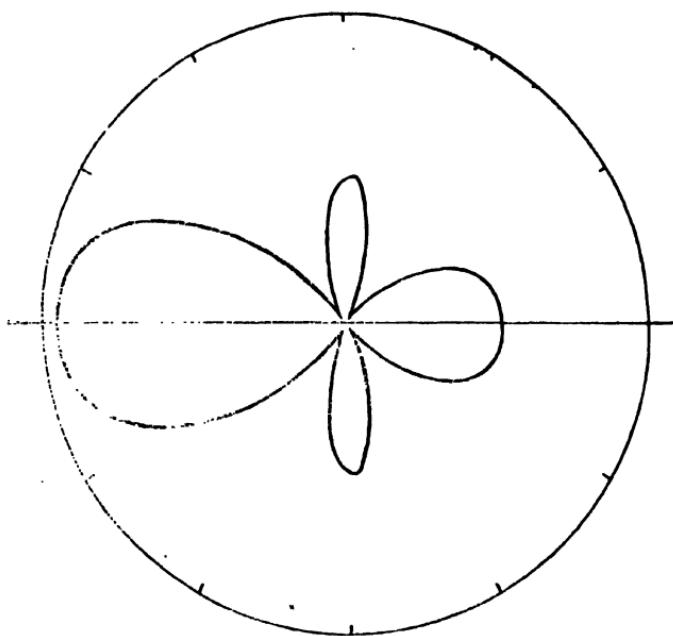
Same as Fig. 3, but with $k(a+b)=3$.

of impact and the case of periodic motion cannot be pushed very far, inasmuch as the fluid motion due to impact is undoubtedly of different character in different directions, and not all throughout the same as in the periodic case.

Now taking $k(a+b) = 2$, we find (neglecting a constant factor) that

$$F = \cdot0992\ P_1\ (\cos\theta) + \cdot2840\ P_2\ (\cos\theta) - \cdot03535\ P_3\ (\cos\theta)$$
$$- \cdot01461\ P_4\ (\cos\theta) + \text{etc.,}$$

$$G = \cdot0496\ P_1\ (\cos\theta) - \cdot4040\ P_2\ (\cos\theta) - \cdot01767\ P_3\ (\cos\theta)$$
$$+ \cdot0315\ P_4\ (\cos\theta) + \text{etc.}\quad(11)$$

The values of F, G and $F^2 + G^2$ for different directions have been calculated and are shown in the following table :—

TABLE I.

Angles (in degrees.)	F × const.	G × const.	$(F^2 + G^2)$ × (const.)
0	+329,000	−338,000	223,144
10	+325,908	−326,525	212,552
20	+297,435	−273,545	162,738
30	+251,537	−214,337	109,300
40	+189,095	−114,659	48,946
50	+114,215	− 24,167	13,572
60	+ 33,341	+ 74,407	6,565
70	− 43,647	+155,306	25,961
80	−107,073	+205,212	53,574
90	−147,250	+213,625	67,405
100	−158,815	+178,920	57,322
110	−140,329	+106,238	30,836
120	− 95,149	+ 8,675	9,106
130	− 34,099	− 99,267	10,957
140	+ 35,677	−202,159	42,100
150	+102,819	−269,237	94,130
160	+157,869	−353,605	150,280
170	+192,648	−392,229	190,913
180	+201,000	−404,000	203,617

Now taking $k(a+b)=3$, we find (neglecting a constant factor) that

$$F = \cdot105P_1\cos\theta + 1\cdot06P_2(\cos\theta) + \cdot016P_3(\cos\theta) - \cdot281P_4(\cos\theta) - \text{etc.}$$

$$G = -\cdot1225P_1(\cos\theta) + \cdot186P_3(\cos\theta) - \cdot024P_4(\cos\theta) - \text{etc.}\quad(12)$$

Fig. 5.
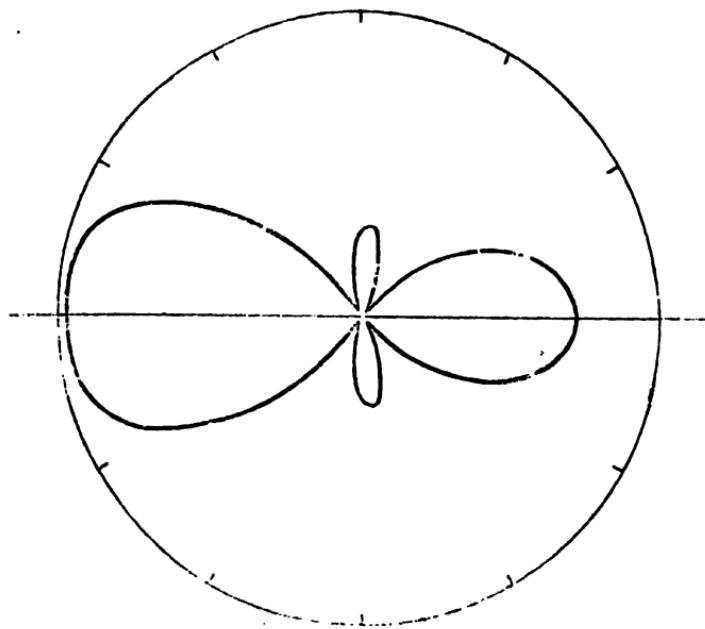
Observed Form of Intensity Curve due to impact
of Spheres of diameters 2 : 1.

(Material, wood; diameters 3 inches and 1½ inches respectively).

The values in different directions have been calculated from these expressions and are shown in the following table :—

TABLE II.

Angles (in degrees.)	F × const.	G × const.	$(F^2 + G^2) \times$ (const.)
0	+900,000	+ 38,000	811,444
10	+890,608	+ 28,804	794,265
20	+849,305	− 2,390	720,805
30	+752,167	− 45,754	567,620
40	+672,469	− 90,446	335,284
50	+309,902	−123,998	111,476
60	− 5,783	−135,346	18,261
70	−313,014	−118,446	111,893
80	−542,728	− 73,554	300,178
90	−635,375	− 9,000	403,306
100	−571,364	+ 60,786	329,762
110	−371,618	+118,638	154,545
120	− 96,799	+149,218	31,610
130	+184,472	+144,494	54,592
140	+412,409	+105,758	180,980
150	+559,907	+ 44,650	315,625
160	+630,625	− 21,410	398,607
170	+654,606	− 69,748	433,925
180	+ 658,000	− 88,000	440,708

The values of $F^2 + G^2$ shown in Tables I and II have been plotted in polar co-ordinates in figs. 3 and 4. It is seen that in both cases, the intensity in the direction of the larger ball is greater than in the direction of the smaller ball. The asymmetry is most marked when $k(a+b)$ has the larger value.

The intensity of the sound in different directions due to the impact of two spheres of wood, diameters 3 inches and $1\frac{1}{2}$ inches respectively, has been measured with the ballistic phonometer and is shown in fig. 5.

It is seen that this curve is intermediate in form between those shown in fig. 3 and fig. 4, exactly as anticipated. The agreement between theory and experiment is thus very striking in this case.

3. *Two spheres of the same diameter but of different materials.*

We have seen in the preceding section that in the expressions F and G for two spheres of the same material but of unequal diameters,

the terms containing the second order zonal harmonic P_2 (cos θ) usually preponderate, and that the intensity diagram is accordingly a curve which consists of four loops. A different result is obtained in the case of two spheres of the same diameter but of markedly unequal densities. The zonal harmonic of the first order preponderates in this case and the intensity diagram is a curve consisting of only two loops. To obtain this result theoretically we have to proceed on exactly the same lines as in the preceding pages.

Taking $a = 1$ inch and $b = 1$ inch, we easily find from the expressions (7) and (8) that

$$2\phi = \left[1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \text{etc.} \right] \frac{P_1 \ (\cos \theta)}{r^2}$$

$$+ 2 \left[1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \text{etc.} \right] \frac{P_2 \ (\cos \theta)}{r^3}$$

$$+ 3 \left[1 - \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{4^5} + \frac{1}{5^5} - \text{etc.} \right] \frac{P_3 \ (\cos. \ \theta)}{r^4}$$

$$+ 4 \left[1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \text{etc.} \right] \frac{P_4 \ (\cos \theta)}{r^5}$$

$$+ \text{etc.,}$$

$$2\phi' = - \left[1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{6^3} - \text{etc.} \right] \frac{P_1 \ (\cos \theta)}{r^2}$$

$$+ 2 \left[1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \text{etc.} \right] \frac{P_2 \ (\cos \theta)}{r^3}$$

$$- 3 \left[1 - \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{4^5} + \frac{1}{5^5} - \text{etc.} \right] \frac{P_3 \ (\cos \theta)}{r^4}$$

$$+ 4 \left[1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \text{etc.} \right] \frac{P_4 \ (\cos \theta)}{r^5}$$

$$- \text{etc.} \tag{13}$$

Summing the series we easily find that the vibrations

$$\left[U_a \frac{\partial \phi}{\partial r} + U_b \frac{\partial \phi'}{\partial r} \right]_{r=2 \text{ inches}}^{ikct}$$

on the surface of the enveloping sphere can be expressed in the form

$$\tfrac{1}{2}[(U_a - U_b) \times \cdot 2254 \ P_1 \ (\cos \theta) + (U_a + U_b) \times \cdot 3550 \ P_2 \ (\cos \theta)$$

$$+ (U_a - U_b) \times \cdot 3645 \ P_3 \ (\cos \theta) + (U_a + U_b) \times \cdot 3080 \ P_4 \ (\cos \theta)$$

$$+ (U_a - U_b) \times \cdot 2325 \ P_5 \ (\cos \theta) + \text{etc.}] \, e^{ikct} \tag{14}$$

Fig. 6.
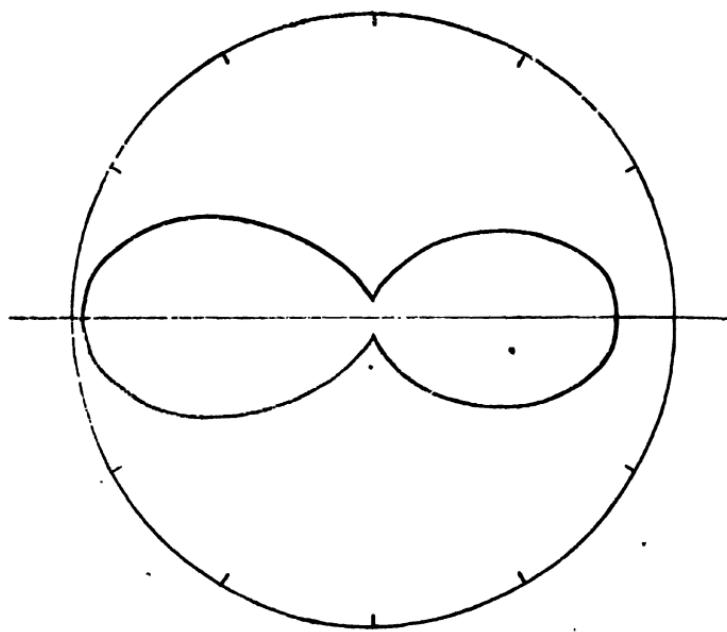
Calculated Form of Intensity Curve due to two
equal Spheres of densities 4 : 1.

If the ball of radius b is four times heavier than the one of radius a, we have

$$U_a = 4\ U_b.$$

So that the vibration on the surface of the enveloping sphere is proportional to the expression

$$[\cdot6762\ P_1(\cos\theta) + 1\cdot7750\ P_2\ (\cos\theta) + 1\cdot0935\ P_3\ (\cos\theta)$$

$$+ 1\cdot5400\ P_4\ (\cos\theta) + \cdot6975\ P_5\ (\cos\theta) + \text{etc.}]\ e^{i k r t}$$

Now taking $k(a+b) = 1$, which will give a wave-length equal to the circumference of the enveloping sphere, we get

$$F = \cdot13524\ P_1\ (\cos\theta) - \cdot04987\ P_2\ (\cos\theta) - \cdot0074\ P_3\ (\cos\theta)$$

$$+ \cdot0007\ P_4\ (\cos\theta) + \text{etc.}$$

$$G = -\cdot0676\ P_1\ (\cos\theta) - \cdot0798\ P_2\ (\cos\theta) + \cdot0047\ P_3\ (\cos\theta)$$

$$+ \cdot0012\ P_4\ (\cos\theta) - \text{etc.}$$

The values of F and G, and of $F^2 + G^2$ in different directions obtained from the preceding expressions are shown in Table III.

TABLE III.

Angles (in degrees.)	F × const.	G × const.	$(F^2 + G^2) \times$ (const.)
0	+ 786,000	−1,415,000	2,620,021
10	+ 793,732	−1,374,897	2,521,061
20	+ 813,819	−1,256,037	2,240,132
30	+ 835,068	−1,068,615	1,839,986
40	+ 845,629	− 826,059	1,397,992
50	+ 828,667	− 549,652	989,741
60	+ 768,690	− 262,257	660,005
70	+ 654,594	+ 7,901	427,780
80	+ 482,433	+ 237,049	288,493
90	+ 252,125	+ 403,500	225,913
100	+ 228,907	+ 495,515	297,466
110	− 331,298	+ 509,107	368,642
120	− 647,986	+ 454,821	626,929
130	− 954,405	+ 347,884	1,031,220
140	−1,229,335	+ 211,923	1,555,385
150	−1,458,496	+ 71,667	2,130,948
160	−1,629,521	− 47,667	2,655,850
170	−1,734,880	− 128,811	3,026,866
180	−1,770,000	− 157,000	3,157,549

Fig. 7.
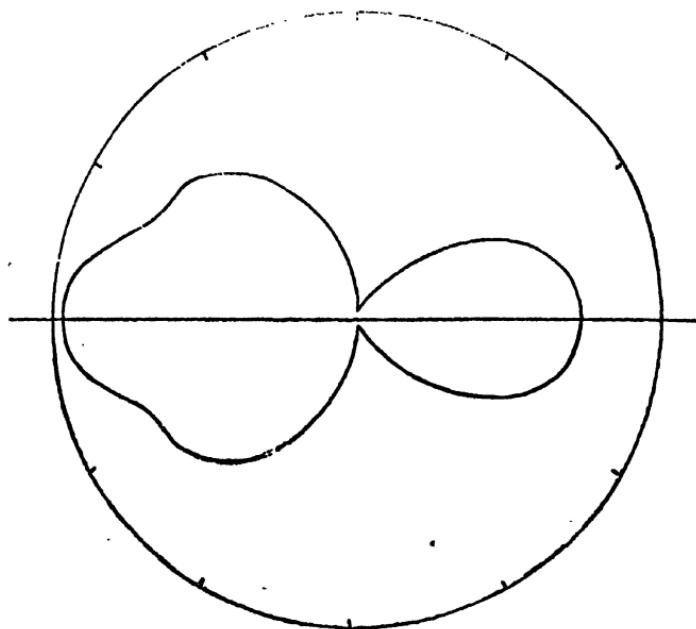
Observed Form of Intensity Curve due to impact of Spheres
of nearly equal diameters but different materials.

Sphere on loft:	Sphere on right :
Material, billiard ball ;	Material, wood ;
Diameter, $2\frac{1}{8}$ inches ;	Diameter, $2\frac{1}{4}$ inches ;
Mass, 150 gms.	Mass, 66 gms.

The values of $(F^2 + G^2)$ shown in Table III have been plotted in polar co-ordinates and are shown in fig. 6. It is seen that the maximum intensity in the direction of the heavier ball is greater than that in the direction of the lighter one.

The experimental curve of intensity of sound due to impact of a sphere of wood, diameter $2\frac{1}{4}$ inch, with a billiard ball of nearly the same size is shown in fig. 7. It is found that the directions of minimum intensity are not quite in the plane perpendicular to the line of impact, they being nearer the side of the lighter ball.

A result of some importance indicated by theory is that when one of the spheres is much heavier than the other, replacing the former by a still heavier sphere of the same diameter should not result in any important alteration in the distribution of the intensity of sound in different directions due to impact. This is clear from expression (14). For when U_a is much larger than U_b, any diminution in the value of U_b should not appreciably affect the value of the expression. This indication of theory is in agreement with experiment. Several series of measurements have been made with various pairs of balls of the same size but of different densities, namely, wood and marble, wood and iron, billiard ball and iron ball and so forth. Generally, similar results are obtained in all cases. It was noticed also that the form of the intensity distribution as shown by the ballistic phonometer was not altogether independent of the thickness of the mica-disk used in the instrument. This is not surprising, as the behaviour of the mica-disk, before the pointer attached to it ceases to touch the mirror of the indicator, would no doubt depend to some extent on the relation between its natural frequency and the frequency of the sound waves set up by impact. The best results have been obtained with a disk neither so thick as to be relatively insensitive, nor so thin as to remain with its pointer in contact with the indicator longer than absolutely necessary.

4. The general case of spheres of any diameter and density.

When the impinging spheres are both of different diameters and of different densities the results generally obtained are that the sound is a maximum on the line of impact in either direction and a minimum which approaches zero in directions asymmetrically situated with reference thereto. Generally speaking no maxima

Fig. 8.
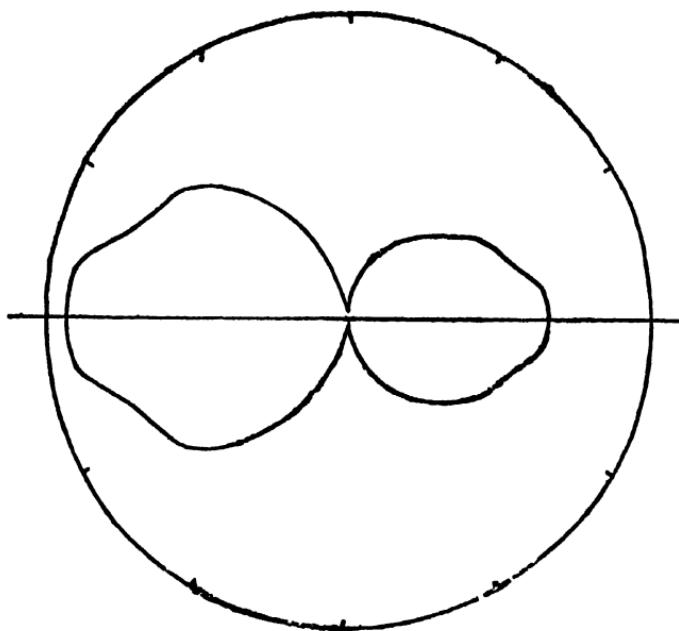
Observed Form of Intensity Curve due to impact of two
Spheres of different diameters and densities.

Sphere on left:	Sphere on right
Material, brass;	Material, wood;
Diameter, 1½ inches;	Diameter, 3 inches
Mass, 118 gms.	Mass, 158 gms.

in lateral directions are noticed, that is, the curve consists of two nearly closed loops. The difference of the intensity of the sound in the two directions of the line of impact may sometimes be considerable. As a typical case, the results obtained by the impact of a sphere of wood 3 inches in diameter with a brass sphere 1½ inches in diameter are shown in fig. 8. It is observed that the sound due to impact is actually of greater intensity on the side of the small brass ball. As a matter of fact the result generally obtained is that the intensity is greater on the side of the ball of the denser material, even if its diameter be smaller.

The mathematical treatment of the general case is precisely on the same lines as in the two preceding sections. It is found in agreement with the experimental result that in practically all cases in which both the densities and the diameters are different, that the zonal harmonic of the first order is of importance and that the intensity curve consists of two nearly closed loops, as in the case of two spheres of the same diameter but of different densities.

5. *Summary and Conclusion.*

The investigation of the origin and character of the sound due to the direct impact of two similar solid spheres which was described in the Phil. Mag. for July, 1916, has been extended in the present paper to the cases in which the impinging spheres are not of the same diameter or material. The relative intensities of the sound in different directions have been measured by the aid of the ballistic phonometer, and in order to exhibit the results in an effective manner, they have been plotted in polar co-ordinates, the point at which the spheres impinge being taken as the origin, and the line of collision as the axis of x. As might be expected, the curves thus drawn show marked asymmetry in respect of plane perpendicular to the line of impact.

A detailed mathematical discussion of the nature of the results to be expected is possible by considering the analogous case of two rigid spheres nearly in contact which vibrate bodily along the line of centres. By choosing an appropriate wave-length for the resulting motion, intensity curves similar to those found experimentally for the case of impact are arrived at. A further confirmation is thus obtained of the hypothesis regarding the origin of the sound suggested by the work of Hertz and of Lord Rayleigh on the theory of elastic impact.

When the impinging spheres, though not equal in size, are of the same or nearly the same density, the intensity curve drawn for the plane of observation shows the sound to be a maximum along the line of impact in either direction, and along two directions making equal acute angles with this line. The sound is a minimum along four directions in the plane.

In practically all other cases, that is when the spheres differ considerably either in density alone, or both in diameter and density, the intensity is found to be a maximum along the line of impact in either direction, and to be a minimum along directions which are nearly but not quite perpendicular to the line of impact. The form of the intensity curve is practically determined by the diameters and the masses of the spheres.

It is hoped when a suitable opportunity arrives to study also the case of oblique impact. The writer has much pleasure in acknowledging the helpful interest taken by Prof. C. V. Raman in the progress of the work described in the present paper.

CALCUTTA,
The 15th of June, 1917.

ON THE VIBRATIONS OF ELASTIC SHELLS PARTLY FILLED WITH LIQUID.

BY

SUDHANSUKUMAR BANERJI.

§ 1. *Introduction.*

The problem considered in this paper is chiefly of acoustical interest in relation to the theory of "musical glasses." This class of instrument consists of a series of thin-walled elastic shells whose gravest modes of vibration are tuned to form a musical scale by partially filling them with a liquid and are excited either by striking or by tangential friction on the rims. The principal features of interest requiring elucidation are (1) the dependance of the pitch of the vibration upon the quantity of liquid contained in the vessel and (2) the mode of vibration of the liquid itself. These features are discussed in this paper for the three cases in which the elastic shell is respectively (1) a hemispherical shell, (2) a cylindrical vessel with a flat bottom and (3) a conical cup, these forms approximating more or less closely to those used in practice. The analytical expressions show that the motion of the liquid is very marked near the margin of the vessel and is almost imperceptible near the centre and also at some depth inside the liquid. This feature becomes more and more marked in the case of the higher modes of vibration of the vessel. Numerical results have also been obtained and tabulated showing the theoretical relation between the quantity of liquid contained in the vessel and the vibration frequency These show that the rapidity with which the frequency falls on addition of liquid is greatest when the vessel is nearly full, this being specially noticeable in the case of the higher modes of vibration.

The general principle of the analytical method used is similar to that adopted by Lord Rayleigh[1] in treating the two-dimensional case of a long cylinder completely filled with liquid which was studied by Auerbach.[2] This case has also been recently discussed by Nikoloi.[3] The lowering of the pitch produced by the liquid is of course due to the added inertia exactly as in the related case of the vibrations of a bar or a string immersed in a liquid which have been studied by Northway,[4] Mackenzie and Kalähne.[5]

Musical glasses are sometimes excited by rotating them about a fixed vertical axis, the tangential friction being produced by a rubber kept in a fixed position. No attempt is made in this paper to consider this somewhat complicated case,[6] which I hope to be able to deal with on a future occasion.

§ 2. *Hemispherical Cups.*

The force which a thin sheet of matter subjected to stress opposes to extension is very great in comparison with that which it opposes to bending. From this Lord Rayleigh concluded that the middle surface of a vibrating shell remains unstretched and proposed a theory[7] of flexural vibrations of curved plates and shells in accordance with this condition. As the direct application of the Kirchhoff-Gehring method led to equations of motion and boundary conditions which were difficult to reconcile with Lord Rayleigh's theory, his

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[1] Lord Rayleigh, *Phil. Mag.*, XV, pp. 385-389, (1883). [*Scientific Papers*, Vol. 2, pp. 208-211].

[2] Auerbach, *Wied. Ann.*, 3, p. 157, (1878) and also *Wied. Ann.*, 17, p. 964, (1882). Reference may also be made to the papers by Montigny, *Bull. del' Acad. de Belg.*, [2], 50, 159, (1880) and by Kolácek, *Wied. Ann.*, 7, 23, (1879) and also *Sitz. Math. Naturw. Cl. Wien*, 87, Abth. 2, (1883).

[3] Nikoloi, *Journ. Russk. Fisik Chimicesk*, 41, 5, pp. 214-227, (1909).

[4] Northway and Mackenzie, *Phys. Rev.*, 13, pp. 145-164, Sept., (1901).

[5] Kalähne, *Ann. d. Physik*, 46, 1, pp. 1-38, (1914).

[6] Reference may be made in this connection to papers by Prof. Love on "The free and forced vibrations of an Elastic Spherical Shell containing a given mass of Liquid," *Proc. Lond. Math. Soc.*, Vol. XIX, where he has studied the case of a rotating spherical shell completely full of liquid, and by Prof. Bryan on "The beats in the vibrations of a revolving cylinder or bell," *Proc. Camb. Phil. Soc.*, Vol. VII, (1892).

[7] Lord Rayleigh, *Proc. Lond. Math. Soc.*, Vol. XIII, p. 4 (1881). See also *Proc. Roy. Soc.*, Vol. 45, p. 45 and 443, (1881); '*Theory of Sound*,' Vol. I, Chap. XA and Love's '*Elasticity*,' Chap. XXXIII.

theory gave rise to much discussion. Later investigations have, however, shown that any extension that may occur must be limited to a region of infinitely small area near the edge of the shell and that the greater part of the shell vibrates according to Lord Rayleigh's type.

Let the radius of the hemisphere be equal to a. Let a point whose natural co-ordinates are a, θ, ϕ be displaced to the position $a+u$, $\theta+v$, $\phi+w$, where u, v, w are to be treated as small. From the condition of inextension

$$(\delta s)^2 = a^2 (\delta\theta)^2 + a^2\sin^2\theta(\delta\phi)^2$$

$$= (a+u)^2(\delta\theta+\delta v)^2 + (\delta u)^2 + (a+u)^2\sin^2(\theta+v)(\delta\phi+\delta w)^2, \quad (1)$$

Lord Rayleigh obtains the three differential equations

$$\frac{\partial v}{\partial \theta} + \frac{u}{a} = 0,$$

$$\frac{\partial v}{\partial \phi} + \sin^2\theta\frac{\partial w}{\partial \theta} = 0,$$

$$\frac{u}{a} + \cot\theta.v + \frac{\partial w}{\partial \phi} = 0,$$

$$(2)$$

which can be integrated in the forms

$$\frac{u}{a} = \frac{\sin m\phi}{-\cos m\phi}\left[A_m(m+\cos\theta)\tan^m\tfrac{1}{2}\theta + B_m(m-\cos\theta)\cot^m\tfrac{1}{2}\theta\right],$$

$$\frac{v}{\sin\theta} = \frac{\sin m\phi}{\cos m\phi}\left[A_m\tan^m\tfrac{1}{2}\theta - B_m\cot^m\tfrac{1}{2}\theta\right],$$

$$(3)$$

$$w = \frac{\cos m\,\phi}{\sin m\,\phi}\left[A_m\tan^m\tfrac{1}{2}\theta + B_m\cot^m\tfrac{1}{2}\theta\right],$$

A_m and B_m being arbitrary constants. These equations determine the character of the displacement of a point in the middle surface.

Since the pole $\theta = 0$ is included, the constant B_m must be considered to vanish and the type of vibration in a principal mode is expressed by the equations

$$u = A_m a(m+\cos\theta)\tan^m\tfrac{1}{2}\theta\sin m\phi,$$

$$v = -A_m\sin\theta\tan^m\tfrac{1}{2}\theta\sin m\phi,$$

$$(4)$$

$$w = A_m\tan^m\tfrac{1}{2}\theta\cos m\phi,$$

in which A_m is proportional to a simple harmonic function of the time.

The potential energy of bending of the vibrating shell is given by

$$V = \frac{8}{3}\pi\mu\frac{\tau^3}{a^2}m^3(m^2-1)^2 A_m^2 \int_c^{\frac{\pi}{2}} \tan^{2m}\frac{1}{2}\theta \frac{d\theta}{\sin^3\theta}$$

$$= \frac{1}{3}\pi\mu\frac{\tau^3}{a^2}(m^3-m)(2m^2-1) A_m^2$$

where $\tau =$ thickness of the shell and $\mu =$ rigidity.

The kinetic energy T is given by the expression

$$T = \frac{1}{2}\pi\sigma a^4\tau\left(\frac{dA_m}{dt}\right)^2 \int_0^{\frac{\pi}{2}} \sin\theta\{2\sin^2\theta + (\cos\theta + m)^2\}\tan^{2m}\frac{1}{2}\theta d\theta$$

$$= \frac{1}{2}\pi\sigma a^4\tau\left(\frac{dA_m}{dt}\right)^2 \int^2 \frac{(2-c)^m}{c^m} [(m-1)^2 + 2(m+1)c - c^2]dc$$

$$= \frac{1}{2}\pi\sigma a^4\tau f(m)\left(\frac{dA_m}{dt}\right)^2, \qquad (6)$$

where σ represents the density of the shell, and

$$f(m) = \int^2 \frac{(2-c)^m}{c^m}[(m-1)^2 + 2(m+1)c - c^2]dc, \qquad (7)$$

which can be evaluated for any integral value of m.

Since the types of vibrations of the shell are entirely determined by the geometry of the middle surface of the shell, it is obvious that the types can under no circumstances be altered by the presence of the liquid in the shell. The liquid gives rise to a surface traction and affects only the arbitrary constant A_m, that is to say, the amplitude and the frequency of vibration of the shell.

The motion of the liquid will depend upon a velocity potential which satisfies the equation

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta} + \frac{\csc^2\theta}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0. \qquad (8)$$

A solution of this differential equation which will correspond to the type of vibration of the shell can be obtained by assuming Φ to be of the form

$$\Phi = \left(C_m r + \frac{D_m}{r^2} \right) \sin m\phi \, . \, \Delta_\theta,$$

where Δ_θ is a function of θ only. Substituting in the differential equation we find that Δ_θ satisfies the equation

$$\frac{d^2 \Delta_\theta}{d\theta^2} + \cot\theta \, \frac{d\Delta_\theta}{d\theta} + (2 - m^2 \mathrm{cosec}^2\theta)\Delta_\theta = 0.$$

The general solution of this differential equation is

$$\Delta_\theta = E_m \tan^m\tfrac{1}{2}\theta \, (m + \cos\theta) + F_m \cot^m\tfrac{1}{2}\theta \, (m - \cos\theta).$$

Neglecting solutions of the type $\cot^m\tfrac{1}{2}\theta \, (m - \cos\theta)$, we see that Φ is of the form

$$\Phi = \left(C_m r + \frac{D_m}{r^2} \right) \sin m\phi \, \tan^m\tfrac{1}{2}\theta(m + \cos\theta), \tag{9}$$

where C_m and D_m are two arbitrary constants.

Let us first take

$$\Phi = C_m \frac{r}{a} \tan^m\tfrac{1}{2}\theta \, (m + \cos\theta) \sin m\phi \cos pt. \tag{10}$$

The relation between C_m and A_m of (4) is readily found by equating the value of $\dfrac{\partial \Phi}{\partial r}$, when $r = a$, to $\dfrac{du}{dt}$, both of which represent the normal velocity at the circumference. We get

$$C_m \cos pt = a^2 \frac{dA_m}{dt}. \tag{11}$$

The expression (10) determines the principal mode of vibration of the liquid. The simple character of the fluid motion as determined by this expression will however be a little disturbed on account of the existence of a free surface and we shall have to add a small

correction to this expression. The condition to be satisfied at the free surface is

$$\frac{\partial^2\Phi}{\partial t^2}+g\,\frac{\partial\Phi}{\partial z}=0, \text{ when } z=h,$$

where h denotes the depth of the liquid surface below the centre of the hemisphere. We shall neglect the force of gravity, inasmuch as the period of free waves of length comparable with the diameter of the shell is much greater than that of the actual motion. The condition to be satisfied at the free surface then becomes simply

$$\Phi=0, \text{ when } z=h.$$

Hence we must have

$$\Phi=C_m\frac{r}{a}\tan^{-1}\tfrac{1}{2}\theta(m-\cos\theta)\sin m\phi\cos pt+f(r,\theta,\phi)\cos pt, \qquad (12)$$

where $f(r, \theta, \phi)$ is a solution of $\nabla^2\Phi=0$ and is such that its differential co-efficient with respect to r vanishes on the spherical boundary and it has the value

$$-C_m\frac{h\sec\theta}{a}\tan^{-1}\tfrac{1}{2}\theta(m+\cos\theta)\sin m\phi \qquad (13)$$

on the free surface

In the particular case, when the shell is completely full of liquid, the differential co-efficient of $f(r, \theta, \phi)$ with respect to r vanishes on the spherical surface and $f(r, \theta, \phi)$ has the value

$$-m\,C_m\frac{r}{a}\sin m\phi \qquad (14)$$

on the surface defined by $\theta=\dfrac{\pi}{2}$.

For the determination of the function $f(r, \theta, \phi)$, spherical harmonics of the complex degree $-\tfrac{1}{2}+p\sqrt{-1}$ are extremely suitable. The properties of these harmonics and their applications to some physical problems have been investigated by Hobson.[1] Solutions of Laplace's equation of the form

$$\frac{1}{\sqrt{r}}\,{\sin\atop\cos}(p\log Ar)\,{\sin\atop\cos}\,m\phi\,K_p{}^m(\cos\theta),$$

[1] Hobson, "On a class of Spherical Harmonics of Complex degree with application to Physical Problems," *Trans. Camb. Phil. Soc.*, Vol. 14 pp. 212-236, (1889).

where $K_p{}^m$ (cos θ) is a harmonic of degree $-\frac{1}{2}+p\sqrt{-1}$ and order m, and is defined by the hypergeometric series

$$K_p{}^m (\cos \theta) = F (m+\tfrac{1}{2}+pi,\ m+\tfrac{1}{2}-pi,\ m+1,\ \sin^2\tfrac{1}{2}\theta),$$

are suitable for our present purpose. These solutions are finite and continuous for all points in the space inside the hemispherical shell (except infinitely near the origin which may be supposed excluded by surrounding it by an infinitely small sphere).

Let us assume

$$f (r, \theta, \phi) = \Sigma_p B_p \sqrt{a} \frac{\sin (p \log r/h)}{\sqrt{r}} \frac{K_p{}^m (\cos \theta)}{K_p{}^m (\cos a)} \sin m\phi, \qquad (15)$$

where $h/a =$ cos a.

Then $\dfrac{\partial}{\partial r} f (r, \theta, \phi) = 0$, when $r=a$, if the p's are the roots of the equation

$$\tan (p \log a/h) - 2p = 0$$

and the summation in the above series extends over all the roots of this equation.

The values of the constants B_p's have to be obtained from the equation

$$-C_m \frac{\cos a}{\cos \theta} \tan^m \tfrac{1}{2}\theta\ (m+\cos \theta) = \Sigma_p B_p \sqrt{a} \frac{\sin [p \log(\sec \theta)]}{\sqrt{h \sec \theta}} \frac{K_p{}^m(\cos\theta)}{K_p{}^m (\cos a)},$$

which must be satisfied for all values of θ between the limits $0 < \theta < a$.

In the particular case, when the shell is completely full of liquid, the values of the constants B_p's can be obtained in a very simple form. Since the origin is a singular point, we exclude the point by surrounding it with a small sphere of radius ϵ, and assume that $f (r, \theta, \phi)$ vanishes on the surface of this sphere. Since in this case $a = \dfrac{\pi}{2}$, we can assume

$$f (r, \theta, \phi) = \Sigma_p B_p \frac{\sqrt{a}\sin\left(p \log\dfrac{r}{\epsilon}\right)}{\sqrt{r}} \frac{K_p{}^m (\cos\theta)}{K_p{}^m (0)} \sin m\phi,$$

where $K_p^m(0) = \dfrac{\sqrt{\pi}\,\Pi(m)}{\Pi(\tfrac{1}{2}m - \tfrac{1}{4} + \tfrac{1}{2}pi)\,\Pi(\tfrac{1}{2}m - \tfrac{1}{4} - \tfrac{1}{2}pi)}$

$$= \frac{\sqrt{\pi}\,\Pi(m)}{\left\{ \dfrac{(2m-1)^2 + p^2}{2^2} \right\} \left\{ \dfrac{(2m-3)^2 + p^2}{2^2} \right\} \cdots}$$

and the summation extends for all values of p which are the roots of the equation

$$\frac{d}{da}\left[\frac{\sin\left(p\log\dfrac{a}{\epsilon}\right)}{\sqrt{a}} \right] = 0,$$

that is to say, the equation

$$\tan\left(p\log\frac{a}{\epsilon}\right) - 2\,p = 0. \tag{11}$$

The constants B_p's have to be determined by the condition that $f(r, \theta, \phi)$ must have the value $-m\,C_m\,\dfrac{r}{a}\sin m\phi$ on the free surface, which is given by $\theta = \dfrac{\pi}{2}$. Hence B_p's are given by

$$-m\,C_m\,\frac{r}{a} = \sum_p \frac{B_p\,\sqrt{a}\sin\left(p\log\dfrac{r}{\epsilon}\right)}{\sqrt{r}}$$

Putting $r = \epsilon.\,e^{\lambda}$ it is easy to see that

$$B_p = -\frac{2\,(p^2 + \tfrac{1}{4})\,m\,C_m}{p^2\log\dfrac{a}{\epsilon} + \tfrac{1}{4}\left(\tfrac{1}{4}\log\dfrac{a}{\epsilon} - 1\right)}\left(\frac{\epsilon}{a}\right)^{\frac{3}{2}}\int_0^{\log\frac{a}{\epsilon}} e^{\frac{3}{2}\lambda}\sin p\lambda\,d\lambda$$

$$= + \frac{8\,(4\,p^2 + 1)\,m\,C_m}{4\,p^2\log\dfrac{a}{\epsilon} + \left(\log\dfrac{a}{\epsilon} - 2\right)}\left[\frac{\sin\left(p\log\dfrac{a}{\epsilon}\right) + p\left(\dfrac{\epsilon}{a}\right)^{\frac{3}{2}}}{9 + 4\,p^2} \right] \tag{1}$$

To obtain an idea of the magnitude of the constant B_p, we shall obtain its value when $\dfrac{a}{\epsilon}$ is a very large quantity. It is easy to see by the method of successive approximation that the roots of the equantion (16) are given by

$$p \log \frac{a}{\epsilon} = X - \frac{\log \dfrac{a}{\epsilon}}{2X} - \left(\log \frac{a}{\epsilon} \right)^2 \left[\frac{1}{4} - \frac{1}{24} \log \frac{a}{\epsilon} \right] \frac{1}{X^3}$$

$$- \left(\log \frac{a}{\epsilon} \right)^3 \left[\frac{1}{4} - \frac{1}{12} \log \frac{a}{\epsilon} + \frac{1}{160} \left(\log \frac{a}{\epsilon} \right)^2 \right] \frac{1}{X^5} - \text{etc.,}$$

where $X = (s+\frac{1}{2})\pi$, s being any integer.

Now, if we take $\dfrac{a}{\epsilon} = 10^5$, the roots are $p_1 = \cdot 321$, $p_2 = \cdot 628$, $p_3 = \cdot 960$, $p_4 = 1\cdot 205$, etc.

Hence we easily find, that the constants Bp_1, Bp_2, Bp_3, etc., have the values $Bp_1 = - \cdot 09\, C_m$, $Bp_2 = - \cdot 08\, C_m$, $Bp_3 = - \cdot 06\, C_m$, etc., from which we infer that the surface correction $f(r, \theta, \phi)$ is a small one. The principal mode of vibration of the liquid is therefore expressed by

$$\Phi = C_m \frac{r}{a} \tan^m \tfrac{1}{2}\theta \; (m + \cos\theta) \sin m\phi \cos pt. \tag{18}$$

If q represent the velocity of the liquid as given by this expression, we have

$$q^2 = \frac{C^2_m}{a^2} \tan^{2m-2} \tfrac{1}{2}\theta \; [(m+\cos\theta)^2 \; (\sin^2 m\phi \tan^2\tfrac{1}{2}\theta + \tfrac{1}{4}m^2 \cos^2 m\phi \sec^2\tfrac{1}{2}\theta)$$

$$+ \{\tfrac{1}{2}m \; (m+\cos\theta) \sec^2\tfrac{1}{2}\theta - \tan\tfrac{1}{2}\theta \sin\theta\}^2 \sin^2 m\phi] \cos^2 pt. \tag{19}$$

Since q is independent of r, the velocity of the liquid at any point in a given radius vector is constant. We see that the velocity varies as $\tan^{m-1}\tfrac{1}{2}\theta$. Hence if we move along any given meridian, the velocity increases from a zero value at the pole at first very slowly then rather abruptly to a large value at the surface, the abruptness of rise being greater the larger the quantity m, that is to say, the higher the mode of vibration of the liquid. Since the velocity of the liquid is constant along any given radius vector, we see that if we consider the motion of the liquid on the surface of

a cone of semi-vertical angle θ, and trace the motion of the liquid as a whole as θ increases, the velocity remains small as θ increases and assumes a large value only at or near the surface. It is obvious therefore that in every case when the cup is not quite filled to the brim, the velocity of the liquid is of very large value near the margin of the vessel and is almost imperceptible near the centre and at some depth in the liquid. In the particular case when the shell is almost filled to the brim, the velocity of the liquid as given by this expression at a point near the centre and also near the free surface is not small. But in this case the free surface correction $f(r, \theta, \phi)$ to the expression for the velocity potential becomes of some importance near the centre and has a sign opposite to it. Consequently the velocity of the liquid near the centre always remains very small. These indications of theory are all confirmed by experiment.

To calculate the kinetic energy of the liquid we have to integrate $\Phi \dfrac{\partial \Phi}{\partial n}$ over the whole boundary of the fluid. At the free surface $\Phi = 0$. We have therefore only to consider the spherical surface.

Therefore

$$T = \tfrac{1}{2}\rho \iint \Phi \frac{\partial \Phi}{\partial n} \, dS$$

$$= \tfrac{1}{2} a\rho \cos^2 pt \int_0^{2\pi} \int_0^{a} [C_m \sin m\phi \tan^m\tfrac{1}{2}\theta \, (m+\cos\theta) + f(a, \theta, \phi)]$$

$$\times C_m \sin m\phi \tan^m\tfrac{1}{2}\theta \, (m+\cos\theta) \sin\theta \, d\theta \, d\phi$$

$$= \frac{\pi}{2} a\rho \cos^2 pt \, C_m^2 \int_0^{a} \tan^{2m}\tfrac{1}{2}\theta \, (m+\cos\theta)^2 \, \sin\theta d\theta$$

$$+ \frac{\pi}{2} a\rho \cos^2 pt \, C_m \sum_r B_r \frac{\sin(p \, \log a/h)}{K_r^m(\cos a)} \int_0^{a} \tan^m\tfrac{1}{2}\theta \, (m+\cos\theta) \, K_r^m(\cos\theta)$$

$$\sin\theta d\theta,$$

ρ being the density of the liquid.   (20)

Since the liquid is supposed to be incompressible, the potential energy is zero.

The sum of the kinetic and potential energies of the solid and liquid together must be independent of the time. Thus we get

$$\left[a^3\rho\int_0^a\tan^{2m}\tfrac{1}{2}\theta(m+\cos\theta)^2\ \sin\theta d\theta+a^5\rho K+a^4\tau\ \sigma f(m)\right]\frac{d^2\,A_m}{dt^2}$$

$$+\tfrac{4}{3}\mu\frac{\tau^3}{a^2}(m^2-m)\ (2m^2-1)A_m=0,$$

where

$$K=\sum_p\frac{B_p}{C_m}\ \frac{\sin\ (p\,\log\,a/h)}{K_p^m\ (\cos a)}\int_0^a\tan^{-\frac{1}{2}}\theta(m+\cos\theta)\ K_p^m\ (\cos\theta)\ \sin\theta d\theta,$$

which is a small quantity.

If A_m varies as $\cos\ (pt+\epsilon)$, we get

$$\left[a^4\rho\int_a^a\tan^{2m}\tfrac{1}{2}\theta(m+\cos\theta)^2\ \sin\theta d\theta+a^4\rho K+\frac{4}{3\pi}M\,f\,(m)\right]p^2$$

$$=\frac{4}{3}\mu\left(\frac{\tau}{a}\right)^3(m^2-m)\ (2m^2-1),$$

where $M=$ mass of the shell.

This equation gives the frequency of vibration of the shell with different quantities of liquid.

If we put

$$F(a,\ m)=\int_{\_}^a\tan^{2m}\tfrac{1}{2}\theta\ (m+\cos\theta)^2\ \sin\,\theta\,d\theta$$

$$=\int_{1+\cos a}^{-}\left(\frac{2-x}{x}\right)^m\ (m-1+x)^2\ dx,$$

the above expression can be written in the form

$$[a\rho\ F\ (a,\ m)+\tau\sigma\,f\,(m)+a\rho K]\,p^2=\frac{4}{3}\ \frac{\mu}{a^3}\ \left(\frac{\tau}{a}\right)^3(m^2-m)(2m^2-1).\ (21)$$

The fall of pitch for the three gravest tones given by $m=2,\ m=3$ and $m=4$ for a brass hemispherical shell of 10 cm. in radius, 2 mm. in thickness and of density 8·6 with different quantities of liquid are shown in Table I.

TABLE I.

| a	Quantity of water in the shell.	$m=2.$			$m=3.$			$m=4.$		
		$F(a, m)$	$f(m)$	$p \times$ const.	$F(a, m)$	$f(m)$	$p \times$ const.	$F(a, m)$	$f(m)$	$p \times$ const.
90°	$\pi a^3 \times ·667$	1·114	1·53	1·80	1·580	1·88	4·63	2·030	2·296	8·76
80°	$\pi a^3 \times ·494$	·570	,,	2·24	·641	,,	6·50	·479	,,	14·51
70°	$\pi a^3 \times ·338$	·291	,,	2·75	·125	,,	9·53	·097	,,	19·42
60°	$\pi a^3 \times ·208$	·123	,,	3·29	·032	,,	10·57	·009	,,	21·45
50°	$\pi a^3 \times ·133$	·058	,,	3·61	·008	,,	10·95	·002	,,	21·64
40°	$\pi a^3 \times ·034$	·026	,,	3·81	·003	,,	11·14	·000	,,	21·68
30°	$\pi a^3 \times ·014$	·015	,,	3·88	·001	,,	11·22	·000	,,	21·69
20°	$\pi a^3 \times ·003$	·013	,,	3·90	·000	,,	11·23	·000	,,	21·69
10°	$\pi a^3 \times ·001$	·011	,,	3·91	·000	,,	11·23	·000	,,	21·69
0°	0	0	,,	3·93	0	,,	11·24	0	,,	21·69
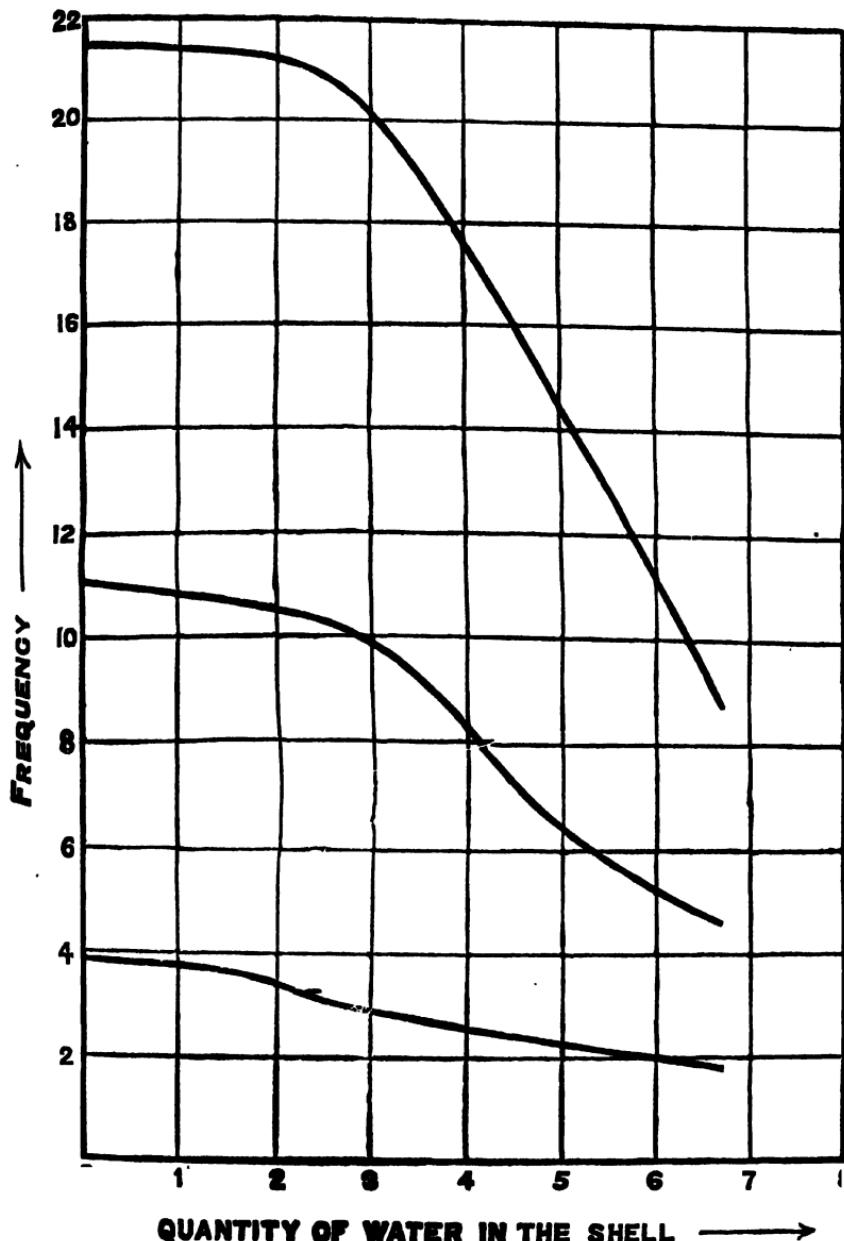
Fig. 1.

In fig. 1, the frequencies have beeen plotted against the quantity of water in the vessel for these three modes of vibrations.

The frequencies of a brass hemispherical shell of about the same radius and thickness loaded with different quantities of water have also been determined experimentally by a photographic method. The results showed a general agreement with the calculated values. As a shell of uniform thickness and of uniform elastic properties throughout could not be procured, and the one that was used was very much deficient in these respects, the slight discrepancy, that was noticed between the calculated and the observed values of the frequency, was probably due to these defects.

§ 3. *Cylindrical Cups.*

The problem of the flexural vibrations of a cylindrical shell is considered in Lord Rayleigh's *Theory of Sound*, Vol. I. § 235 c. If the displacements at any point a, θ, z of the cylinder be δr, $a\delta\theta$, δz, then

$$\delta r = -n\,(A_n a + B_n z)\,\sin n\theta,$$
$$a\delta\theta = (A_n a + B_n z)\,\cos n\theta, \qquad (22)$$
$$\delta z = -n\,B_n a\,\sin n\theta,$$

Supposing now that the cup has been formed by an inextensible disk being attached to the cylinder at $z=0$, the displacements δr, $a\delta\theta$ must vanish for that value of z. Hence $A_n = 0$, and

$$\delta r = -n\,B_n z\,\sin n\theta,\quad a\delta\theta = B_n z\,\cos n\theta,\quad \delta z = -n\,B_n a\,\sin n\theta, \qquad (23)$$

the constant B_n is proportional to a simple harmonic function of the time, say, $\cos pt$.

Since the displacements δr and $a\delta\theta$ are proportional to z and the displacement δz is independent of z, it is obvious, that when z is large the displacements δr and $a\delta\theta$ are also very large compared to δz, that is to say, near the free end of the shell, the displacement δz is very small compared to δr or $a\delta\theta$. But at the bottom of the shell, the displacements δr and $a\delta\theta$ vanish and δz remains constant. We conclude, therefore, from the law of continuity that the disk at the bottom must have a small normal vibration. If w denote the normal

displacement of the disk, it is well-known that w satisfies the differential equation

$$\frac{\partial^2 w}{\partial t^2} + c^4 \nabla^4 w = 0, \qquad (24)$$

where $\nabla^2 = \dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} + \dfrac{1}{r^2}\dfrac{d^2}{d\theta^2}$ and c is a certain constant.

If $w \propto \cos (pt+\epsilon)$, then the equation becomes

$$\nabla^4 w - \nu^4 w = 0,$$

where $\nu^4 = \dfrac{p^2}{c^4}$.

A solution of this differential equation is known to be

$$w = C_n \, J_n \, (\nu r) \sin \, n\theta.$$

Hence we shall take

$$w = C_n \, J_n \, (\nu r) \sin \, n\theta \cos \, pt. \qquad (25)$$

The value of the constant C_n can be obtained from the condition that w and δz must be continuous at the boundary. This gives

$$C_n \cos pt = -\frac{B_n}{n} \frac{a}{J_n \, (\nu a)}. \qquad (26)$$

We can assume that $J_n(\nu a)$ which depends on ν, is very large, and consequently that the normal vibration of the disk is very small. The potential energy of deformation for a length l of the cylinder is

$$V = \frac{4\pi\mu\tau^3}{3a} \, (n^2 - 1)^2 \left[\frac{\lambda+\mu}{\lambda+2\mu}\frac{n^2 l^2}{3a^2} + 1\right] B_n^2. \qquad (27)$$

The potential energy of vibration of the disk is given by

$$\frac{E\tau'^3}{3(1-\mu^2)}\int_0^a \int_0^{2\pi} \left[(\nabla^2 w)^2 - 2\,(1-\mu) \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial \cdot \partial y}\right)^2 \right\} \right] r d\theta dr,$$

where $E =$ Young's modulus and $\tau' =$ thickness of the disk. The value of this integral can be easily obtained. But as we regard the vibration of the disk compared to that of the cylindrical surface to be very small, the value of this expression is also very small. We can denote this expression by $V_1 B_n^2$, since w has been shown to be proportional to B_n.

If the volume density be σ, we get the expression for the kinetic energy in the form

$$T = \tfrac{1}{4}\pi\sigma \ rla[\tfrac{1}{3}l^2 \ (1+n^2)+n^{-2} \ a^2]\left(\frac{dB_x}{dt}\right)^2$$

$$+ \ \tfrac{1}{2}\pi\sigma r' \ \frac{a^2}{n^2[J_n \ (va)]^2} \ \left(\frac{dB_x}{dt}\right)^2 \int\limits_{-}^{a}[J_n \ (vr)]^2 \ rdr.$$

(28)

If the cylinder contains frictionless incompressible fluid, the motion of the fluid will depend upon a velocity potential Φ which satisfies the equation $\nabla^2 \ \Phi=0$, or in cylindrical co-ordinates

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r} +\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\partial^2\Phi}{\partial z^2}=0.$$

The solution of this differential equation can be written in either of the forms

$$\Phi=a_n \ zr^n \ \sin n\theta \ \cos pt,$$

(29)

$$\Phi=\beta_n \ e^{-kz} \ J_n \ (kr) \ \sin n\theta \ \cos pt.$$

(30)

The boundary conditions to be satisfied by Φ are

$$(i) \quad \frac{\partial\Phi}{\partial r} =\frac{d\delta r}{dt}, \text{ when } r=a.$$

(31)

$$(ii) \quad \frac{\partial\Phi}{\partial z} =\frac{dw}{dt}, \text{ when } z=0.$$

(iii) $\Phi=0$ at the free surface, i.e., when $z=h$.

We assume that

$$\Phi=a_n \ zr^n \ \sin n\theta \ \cos pt$$

$$+\sum_k J_n \ (kr) \ [D_k \cosh kz+E_k \ \sinh kz] \ \sin n\theta \ \cos pt,$$

(32)

where the summation extends for all values of k which are roots of the equation

$$\frac{\partial}{\partial a}J_n \ (ka)=0.$$

(33)

We at once get by condition (i),

$$a_n \cos pt = -\frac{1}{a^{n-1}} \frac{dB_n}{dt}. \tag{34}$$

The condition (ii) gives

$$-\left[\frac{a \, J_n(vr)}{n \, J_n(va)} - \frac{r^n}{a^{n-1}}\right] \frac{1}{k} \frac{dB_n}{dt} = \sum_k E_k J_n(kr) \cos pt.$$

This equation must be satisfied for all values of r between the limits $(0 < r < a)$ and will give the value of the constant E_k.

Now since

$$\int_0^a r^{n+1} J_n(kr) \, dr = \frac{na^n}{k^2} J_n(ka),$$

$$\int_0^a J_n(kr) J_n(vr) \, r dr = \frac{av}{k^2 - v^2} J_n(ka) J_n'(va),$$

and

$$\int_0^a J_n^2(kr) \, r dr = \tfrac{1}{2} a^2 \left(1 - \frac{n^2}{k^2 a^2}\right) J_n^2(ka),$$

we get

$$-\frac{1}{k} \frac{dB_n}{dt} \left[\frac{a}{n J_n(va)} \int_0^a J_n(kr) J_n(vr) \, r dr - \frac{1}{a^{n-1}} \int_0^a r^{n+1} J_n(kr) \, dr\right]$$

$$= E_k \cos pt \int_0^a J_n^2(kr) \, r dr,$$

and therefore

$$E_k \cos pt = -\frac{2a}{ka} \frac{dB_n}{dt} \frac{k^2 a^2}{(k^2 a^2 - n^2) J_n(ka)} \left[\frac{v}{(k^2 - v^2)n} \frac{J_n'(va)}{J_n(va)} - \frac{na}{k^2 a^2}\right]. \tag{35}$$

The condition (iii) gives

$$a_n h r^n + \sum_k (D_k \cosh kh + E_k \sinh kh) J_n(kr) = 0.$$

for all values of r between the limits $(0 < r < a)$. Therefore we get

$$D_k \cosh kh + E_k \sinh kh = -a_n \frac{2na^n h}{(k^2 a^2 - n^2) J_n(ka)} \tag{36}$$

The equations (35) and (36) give the values of constants D_k and E_k.

To calculate the kinetic energy we have to integrate $\Phi \dfrac{\partial \dot{\psi}}{\partial n}$ over the boundary of the shell. At the free surface $\dot{\Phi}=0$. We have therefore only to consider the cylindrical surface and the bottom. The expression can be written in the form

$$T=\tfrac{1}{2}\pi\rho\,\cos^2 pt\left[na_n{}^2 a^2{}^n\frac{h^3}{3}+na_n a^n \Sigma J_n(ka)\left\{D_k\left(\frac{h\sinh kh}{}\right.\right.\right.$$

$$\left.-\frac{1}{k^2}\cosh kh+\frac{1}{k^3}\right)+E_k\left(\frac{h\cosh kh}{k}-\frac{1}{k^2}\sinh lh\right.$$

$$\left.\left.-\Sigma_k\left\{a_n D_k\frac{na^2}{k^2}J_n(ka)+\tfrac{1}{2}ka^2 E_k D_k\left(1-\frac{n^2}{k^2 a^2}\right)J_n{}^2(ka)\right\}\right]. \quad (37)$$

The constants E_k and D_k are very small compared to a_n. If we neglect E_k and D_k, the expression for the kinetic energy reduces to the simple form

$$T=\frac{\pi}{2}\rho na_n{}^2 a^2{}^n\frac{h^3}{3}\cos^2 pt. \quad (38)$$

In this case the expression for Φ reduces to the form

$$\Phi=a_n z r^n\sin n\theta\,\cos pt. \quad (39)$$

This expression represents the principal mode of vibration of the liquid and all the other coexistent modes are very small compared to this one. Since the expression for the velocity varies as r^{n-1}, the velocity is very marked near the margin of the vessel and is almost imperceptible near the centre. The sum of the kinetic and potential energies of the solid and liquid together must be independent of the time. From this we easily obtain an expression for the frequency of vibrations in the most general case from the expressions for the kinetic and potential energies already given. If we neglect E_k and D_k, the frequency equation takes a very simple form. The expression in this case is

$$\left[\sigma\pi la\left\{\tfrac{1}{8}l^2(1+n^2)+n^{-2}a^2\right\}+\tfrac{1}{3}\rho na^2 h^3\right]p^2$$

$$=\frac{8\mu\tau^2 l}{3a}(n^2-1)^2\left[\frac{\lambda+\mu}{\lambda+2\mu}\frac{n^2 l^2}{3a^2}+1\right]. \quad (40)$$

Thus we see that the law of variation of the frequency with the height of water in the vessel can be expressed in the form

$$p^2 = \frac{1}{A + B.\left(\dfrac{h}{l}\right)^3}$$

where A and B are two constants for the vessel.

For a glass cylinder whose dimensions are given by $l/a = 4$, $\tau/a = \cdot 02$ and which has the density $\sigma = 2\cdot 6$ and the elastic constants $\mu = 1\cdot 8$ and $\lambda = 1\cdot 53$, we easily find that the frequency p_n is given by

$$\left[\cdot 052 \left\{ (1+n^2) + \frac{\cdot 1875}{n^2} \right\} + n \left(\frac{h}{l}\right)^3 \right] p_n{}^2$$

$$= \frac{8\mu\tau^3}{3a^2 l^2} (n^2-1)^2 \left[10\cdot 4n^2 + 3 \right].$$

For the three gravest tones given by $n=2$, $n=3$ and $n=4$, the values of the frequencies p_2, p_3 and p_4 with different quantities of water in the cylinder are shown in Table II.

TABLE II.

h/l	$p_2 \times$ const.	$p_3 \times$ const.	$p_4 \times$ const.
0	12·47	34·44	65·63
·1	12·33	34·40	65·48
·2	12·02	33·98	64·47
·3	11·29	32·05	61·95
·4	10·15	29·45	57·79
·5	8·85	26·26	52·45
·6	7·61	22·99	46·68
·7	6·56	19·97	41·09
·8	5·60	17·34	36·05
·9	4·97	15·13	31·67
1·0	4·21	13·25	27·93

The values of the frequencies given in Table II have been plotted in fig. 2.

§ 4. Conical Cups.

It is shown in Lord Rayleigh's *Theory of Sound* in the article already referred to that if a cone for which $\rho = \tan\gamma.z$, γ being the
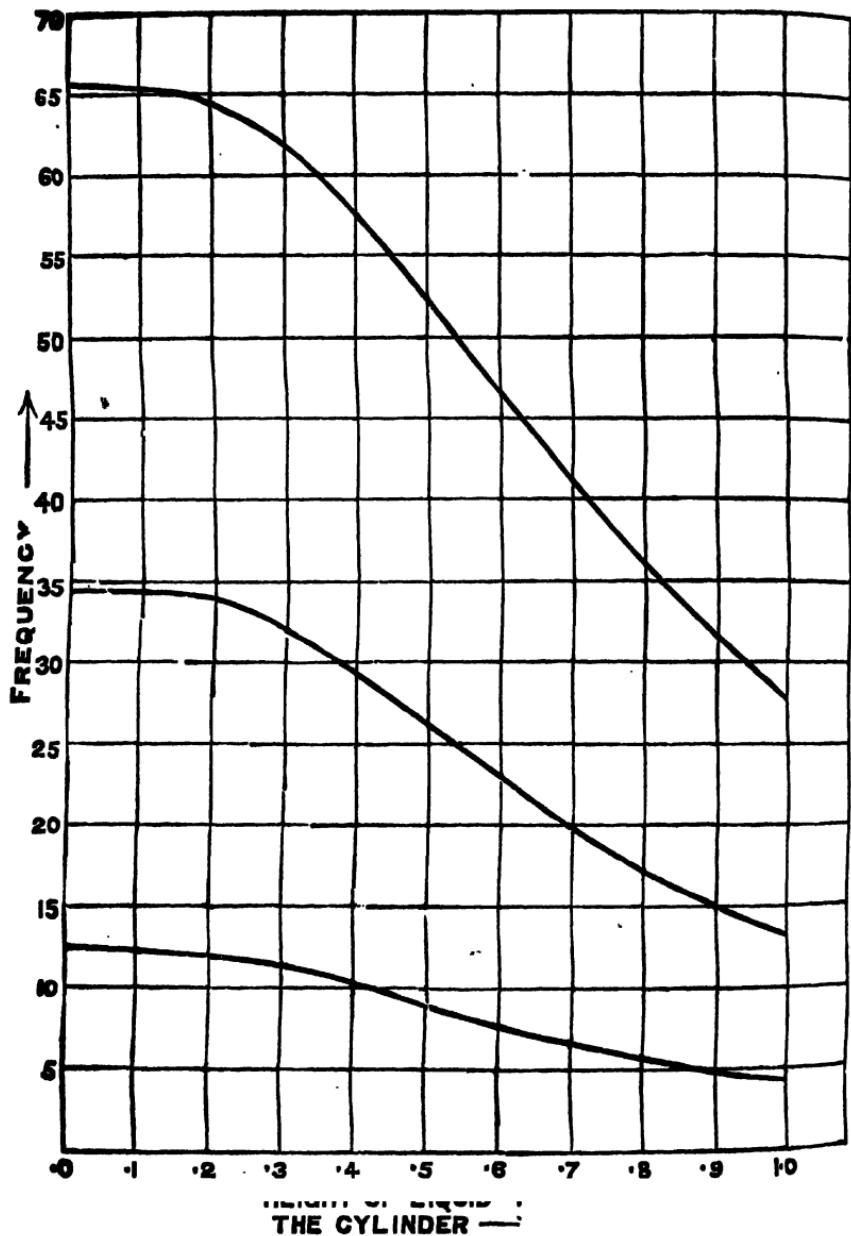
Fig. 2,

semi-vertical angle, executes flexural vibrations, the displacements $\delta\rho$, $\delta\phi$, δz at any point whose cylindrical coordinates are (ρ, ϕ, z) are given by

$$\delta\rho = n \tan\gamma \, (A_n z + B_n) \sin n\phi,$$

$$\delta\phi = (A_n + B_n z^{-1}) \cos n\phi, \qquad (41)$$

$$\delta z = \tan^2\gamma \, [n^{-1} B_n - n (A_n z + B_n)] \sin n\phi.$$

If the cone be complete up to the vertex at $z = 0$, then $B_n = 0$, so that

$$\delta\rho = n \tan\gamma . A_n \, z \sin n\phi,$$

$$\delta\phi = A_n \cos n\phi, \qquad (42)$$

$$\delta z = -n \, A_n \tan^2\gamma . z \sin n\phi.$$

If the displacements in polar coordinates $(r, \ , \phi)$ be denoted by δr, $\delta\theta$, $\delta\phi$, we easily obtain

$$\delta\phi = A_n \cos n\phi,$$

$$\delta r = \delta\rho \sin \gamma + \delta z \cos\gamma = 0, \qquad (43)$$

$$r\delta\theta = \delta\rho \cos\gamma - \delta z \sin\gamma = n \, A_n \tan\gamma . r \sin n\phi.$$

It is easy to see that the potential energy of deformation for a length l of the cone

$$W = \frac{4\pi}{3} \mu\tau^3 \, \frac{\lambda+\mu}{\lambda+2\mu} A_n^2 \sin\gamma \left[\left(-n^2 \frac{\tan\gamma}{\sin^2\gamma} + n \tan\gamma + n \cot\gamma\right)^2 \right.$$
$$\left. + \cos^2\gamma \right] \log\frac{2l}{\tau}$$

where $\tau = $ thickness.* (44)

The expression for the kinetic energy of vibration of the shell can be easily obtained in the form

$$T = \frac{\pi}{8} \sigma\tau l^4 \sin^3\gamma \, [n^2 \sec^2\gamma + 1]\left(\frac{dA_n}{dt}\right)^2. \qquad (45)$$

* This equation can be readily deduced from a very general expression for the potential energy due to strain in curvilinear coordinates obtained by Prof. Love. (*Vide* his paper on "The small free vibrations and deformation of a thin Elastic Shell," *Phil. Trans.*, Vol. 179, 1888, A.) The expression has been criticised by Prof. Basset (*Phil. Trans.*, Vol. 181, 1890, A) on the ground that Prof. Love has omitted several terms which involve the extension of the middle surface. As the inextensional vibrations only have been considered in this paper, this criticism does not affect us in any way.

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If the cup contains frictionless incompressible fluid, the velocity potential of the fluid must satisfy Laplace's equation. Let us assume that the velocity potential is given by

$$\Phi = C_n r^2 \phi_n (\cos\theta) \sin n\phi . \cos pt, \tag{46}$$

where $\phi_n (\cos\theta)$ is a function of θ only.

Then it is easy to see by substitution in the differential equation

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta} + \frac{\operatorname{cosec}^2\theta}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0,$$

that $\phi_n (\cos\theta)$ satisfies the equation

$$\frac{d^2\phi_n}{d\theta^2} + \cot\theta \frac{d\phi_n}{d\theta} + (6 - n^2 \operatorname{cosec}^2\theta)\,\phi_n = 0.$$

A solution of this differential equation can be easily obtained in the form

$$\phi_n(\cos\theta) = \tan^n\tfrac{1}{2}\theta \left[(1-n)(2-n) - 6(2-n)\cos^2\frac{\theta}{2} + 12\cos^4\frac{\theta}{2} \right]. \tag{47}$$

The relation between C_n and A_n can be easily obtained by equating the value of $\dfrac{\partial\Phi}{r\,\partial\theta}$, when $\theta = \gamma$, to $\dfrac{d(r\delta\theta)}{dt}$, both of which represent the normal velocity at the boundary. We thus get

$$C_n\cos pt\,\frac{\partial}{\partial\gamma}\left[\tan^n\tfrac{1}{2}\gamma\left\{(1-n)\,(2-n) - 6\,(2-n)\cos^2\frac{\gamma}{2} + 12\cos^4\frac{\gamma}{2}\right\}\right]$$

$$= n\tan\gamma\frac{dA_n}{dt}. \tag{48}$$

The principal mode of vibration of the liquid will therefore be expressed by (46) except for a small correction to be introduced on account of the existence of a free surface. At the free surface the condition to be satisfied is given by

$$\Phi = 0 \text{ when } z = h,$$

where h denotes the height of the liquid.

To satisfy this condition, we take

$$\Phi = C_n r^2 \phi_n(\cos\theta) \sin n\phi \cos pt + \sum_m D_m r^m P_m^n (\cos\theta) \sin n\phi \cos pt, \tag{49}$$

where the summation extends for all values of m which are the roots of the equation

$$\frac{d}{d\gamma} P_n^n (\cos\gamma) = 0. \tag{50}$$
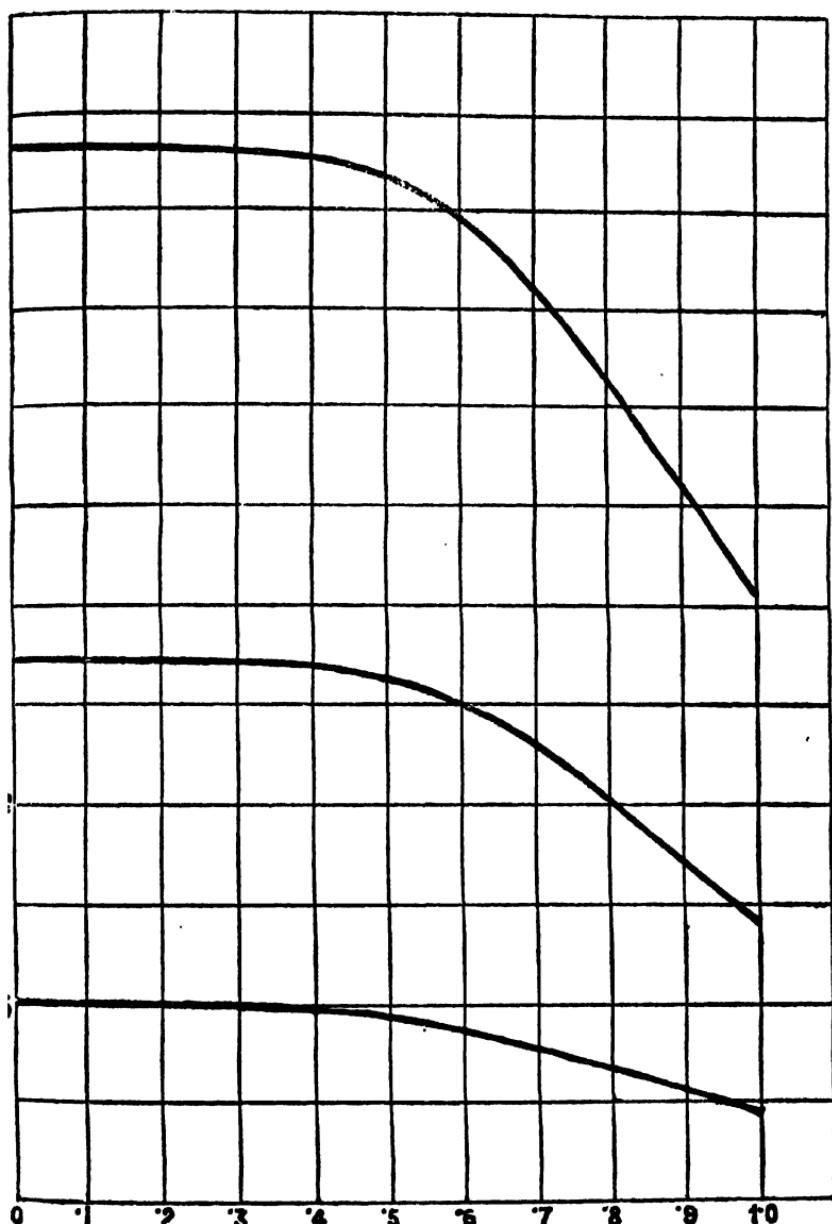
THE RATIO OF THE SLANT HEIGHT OF WATER
TO THAT OF THE CONE ⟶

Fig. 3.

The constants D_n's have to be determined by means of the equation

$$C_n(h\sec\theta)^2\phi_n(\cos\theta) + \sum D_m(h\sec\theta)^m P_m{}^n(\cos\theta) = 0, \qquad (51)$$

which must be satisfied for all values of θ between the limits $0 < \theta < \gamma$. Approximate values of the constants D_m's can be easily obtained from this equation. To get an idea of the magnitude of the constant D_m, we shall obtain its value in the particular case when the semi-vertical angle γ of the cone is small and the height h of the liquid is large compared to the radius of the cross-section of the cone by the free surface. In this case the free surface can be taken to be practically coincident with the surface of the sphere $r = h$. The equation for determining D_m is then

$$C_n h^2 \phi_n(\cos\theta) + \sum D_m h^m P_m{}^n(\cos\theta) = 0,$$

Now, since

$$\int_{\cos\gamma}^{1} P_m{}^n(\cos\theta) P_{m'}{}'^n(\cos\theta) \sin\theta d\theta = 0,$$

m, m' being two different roots of the equation (50), and

$$\int_{\cos\gamma}[P_m{}^n(\cos\theta)]^2 \sin\theta \, d\theta = \frac{1-\cos^2\gamma}{2m+1} P_m{}^n(\cos\gamma) \frac{\partial^2}{\partial m \, \partial\cos\gamma} P_m{}^n(\cos\gamma),$$

we easily get

$$D_n = -\frac{2m+1.}{1-\cos^2\gamma \, P_m{}^n(\cos\gamma)} \cdot \frac{C_n h^{2-n}\int_{\cos\gamma} \phi_n(\cos\theta) P_m{}^n(\cos\theta)\sin\theta d\theta}{\dfrac{\partial^2}{\partial m \, \partial\cos\gamma} P_m{}^n(\cos\gamma)}$$

Neglecting the small correction introduced by the free surface, we see that the kinetic energy of the fluid motion is

$$\frac{\pi}{2} \rho \, C_n{}^2 \cos^2 pt \, \phi_n(\cos\gamma) \frac{\partial\phi_n(\cos\gamma)}{\partial\gamma} \sin\gamma \frac{h^5}{5} \sec^5\gamma.$$

Since the sum of the kinetic and potential energies of the solid and liquid together must be independent of the time, we easily

obtain, on assuming that $A_n \propto \cos pt$, the frequency equation in the form

$$\left[\frac{1}{4} \sigma \tau l^4 \sin^3\gamma \, (n^2 \sec^2\gamma + 1) + \rho n^2 \tan^2\gamma \sin\gamma \, \frac{\phi_n(\cos\gamma)}{\dfrac{\partial}{\partial\gamma} \phi_n(\cos\gamma)} \frac{H^5}{5} \right] p^2$$

$$= \frac{8}{3} \mu \tau^3 \frac{\lambda+\mu}{\lambda+2\mu} A_n{}^2 \sin\gamma \left[\left(-n^3 \frac{\tan\gamma}{\sin^2\gamma} + n \tan\gamma + n \cot\gamma \right)^2 \right.$$

$$\left. + \cos^2\gamma \right] \log \frac{2l}{\tau},$$

where $H = h \sec\gamma$, H being the slant height of the liquid.

In this case we see that the law of variation of frequency with the height of liquid can be expressed in the form

$$p^2 = \frac{1}{A + B \left(\dfrac{h}{l} \right)}$$

A and B being two constants for the particular shell. The frequencies p_2, p_3 and p_4 with different quantities of water for the three gravest modes of vibrations given by $n=2$, $n=3$ and $n=4$ have been calculated from this expression for a cone of semi-vertical angle 30°, the ratio of the thickness of the sides of the cone to the slant height being equal to ·02 and are shown in Table III.

TABLE III.

H/l	$p_2 \times$ const.	$p_3 \times$ const.	$p_4 \times$ const.
0	5·030	13·58	26·75
·1	5·030	13·58	26·75
·2	5·028	13·57	26·73
·3	5·008	13·54	26·69
·4	4·937	13·41	26·47
·5	4·761	13·08	25·94
·6	4·428	12·43	24·87
·7	3·942	11·64	23·13
·8	3·399	10·07	20·79
·9	2·808	8·63	18·07
1·0	2·310	7·26	15·41

The curves showing the fall of frequency for these three modes of vibrations of the cone when loaded with different quantities of water are plotted in fig. 3.

CALCUTTA, *the 30th July, 1918.*

ON THE NUMERICAL CALCULATION OF THE ROOTS OF THE EQUATIONS $P_n^m(\mu)=0$ AND $\dfrac{d}{d\mu} P_n^m(\mu)=0$ REGARDED AS EQUATIONS IN n.

BY

BHOLANATH PAL.

§ 1.

The object of the present paper is to obtain expressions for n which make the functions $P_n^m(\mu)$ and $\dfrac{d}{d\mu} P_n^m(\mu)$ vanish by the use of an asymptotic expansion for $P_n^m(\mu)$ recently obtained by Dr. Watson.[*] A generalisation of Laplace's formula

$$P_n^m(\cos\theta) = \sqrt{\frac{2}{\pi}} \frac{\Gamma(n+m+1)}{\Gamma(n+\frac{3}{2})} \left[\frac{\cos\{(n+\frac{1}{2})\theta - \frac{1}{4}\pi + \frac{1}{2}\}}{(2\sin\theta)^{\frac{1}{2}}} \right. $$
$$ \left. + \frac{1^2 - 4m^2}{2(2n+3)} \frac{\cos\{(n+\frac{3}{2})\theta - \frac{3}{4}\pi + \frac{1}{2}m\pi\}}{(2\sin\theta)^{\frac{3}{2}}} + \ldots\ldots \right]$$

obtained by Prof. Hobson,[†] affords also a convenient expression convergent when $\frac{1}{6}\pi < \theta < \frac{5}{6}\pi$ and asymptotic over the range $0 < \theta < \pi$ for the determination of n. This expression has been used by Macdonald[‡] to obtain an expression for n which makes the function $P_n^m(\mu)$ vanish. It will be shown in this paper that Watson's asymptotic expansion affords a much more simpler expression for n.

The roots of these equations are of very great importance in a number of physical problems involving a conical boundary. For example, it has been shown by Prof. Carslaw[§] that the scattering

* Watson, " Asymptotic expansions of hypergeometric functions," *Trans. Camb. Phil. Soc.*, Vol. XXII, No. 14, pp. 277-308, (Oct. 1918).

† Hobson, *Philosophical Transactions* (1896) p. 486.

‡ Macdonald, *Proc. Lond. Math. Soc.* (1899).

§ Carslaw, *Math. Annalen*, (Feb., 1914). See also a paper by the same author in *Phil. Mag.*, 1910,

of sound waves due to a source $\dfrac{e^{-iR}}{R}$ situated at a point (r', o, o)

by a cone $(\theta = \theta_0)$ is given by

$$u = \frac{4}{\sqrt{rr'}} \sum_n e^{\frac{1}{2}(n+\frac{1}{2})i\pi} (n+\tfrac{1}{2}) \, K_{n+\frac{1}{2}}(ikr') \, J_{n+\frac{1}{2}}(kr)$$

$$\frac{P_n(\mu)}{(1-\mu_o^2)\, P_n(\mu_o)\, \dfrac{d^2}{dnd\mu_o}\, P_n(\mu_o)}$$

for $r < r'$, the summation being taken over the zeroes greater than $-\frac{1}{2}$

of $\dfrac{d}{d\mu_0} P_n(\mu_o)$. As another example, I may mention that the

problem* of the determination of the oscillations of a sea bounded

by parallels of latitude requires the calculation of the roots of the

equation $\dfrac{d}{d\mu} P_n^m(\mu) = 0$ regarded as an equation in n.

As the writer wishes to apply the results obtained in this paper

to the problem of the cone solved by Carslaw,† the numerical values

of the first few roots of the equation $\dfrac{d}{d\mu} P_n^m(\mu) = 0$ when $\theta = \dfrac{\pi}{4}$

have been obtained and are shown in Tables I, II and III. Further

tables for the roots of the equations $P_n^m(\mu) = 0$ and $\dfrac{d}{d\mu} P_n^m(\mu) = 0$

for the cases when $\theta = \frac{1}{12}\pi, \frac{1}{6}\pi$ and $\frac{1}{4}\pi$ are in preparation and will

be given in a subsequent paper.

My best thanks are due to Dr. S. K. Banerjee for having

suggested this work to me and for the help which I have received from

him in the preparation of this paper.

§ 2.

The asymptotic expansion of $P_n^m(\cos\theta)$ when $|n|$ is large obtained

by Watson can be written in the form

$$P_n^m(\cos\theta) = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} \frac{1}{\sqrt{(2n\pi \sin\theta)}} \left[e^{\left\{(n+\frac{1}{2})\theta + \frac{m\pi}{2} - \frac{\pi}{4}\right\}i} \right.$$

$$\left. \sum_{s=0}^{\infty} \frac{c_s \Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2})n^s} + e^{-\left\{(n+\frac{1}{2})\theta + \frac{m\pi}{2} - \frac{\pi}{4}\right\}i} \sum_{s=0}^{\infty} \frac{c'_s \Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2})n^s} \right], \quad (1)$$

* Vide Lamb's *Hydrodynamics*, §200, p. 292 (Third Edition).

† *Ibid.*

where c_s is equal to $(1-e^{2x})^s$ multiplied by the co-efficient of T^{2s} in the expansion of

$$\left\{1-\frac{T}{1-e^x}\right\}^m \left\{1+\frac{T}{1+e^x}\right\}^{-m} \left\{1+T\operatorname{cosech} x\right\}^{-1}$$

$$\left[\frac{1-e^x}{T^2} \log\left\{\frac{T^2}{1-e^{2x}-2e^xT-T^2}\right\}\right]^{-s-\frac{1}{2}}$$

in ascending powers of T and c'_s is obtained by changing the sign of x in c_s and $x=i\ \theta$.

From this we find $c_o=1$, $c'_o=1$,

$$c_1 = m^2-\tfrac{1}{2}+i\ (m^2-\tfrac{1}{4})\cot\theta,$$

$$c'_1 = m^2-\tfrac{1}{2}-i\ (m^2-\tfrac{1}{4})\cot\theta,$$

$$c_2 = C_2+i\ C_2',$$

$$c'_2 = C_2-i\ C'_2,$$

where

$$C_2 = \tfrac{1}{6}\ (m^2-\tfrac{1}{2})^2-\tfrac{1}{6}\ (m^2-\tfrac{9}{4})\ (m^2-\tfrac{1}{4})\cot{}^2\theta,$$

$$C_2' = \tfrac{1}{3}\ (m^2-\tfrac{3}{2})\ (m^2-\tfrac{1}{4})\cot\theta.$$

Substituting these values of c_o, c_1......in (1) and equating the real and imaginary parts, we get

$$P_n{}^m(\cos\theta) = \frac{\Gamma(n+1)}{\Gamma(n-m+1)}\left(\frac{2}{n\pi\sin\theta}\right)^{\frac{1}{2}}\left[\cos\left\{(n+\tfrac{1}{2})\theta+\frac{m\pi}{2}-\frac{\pi}{4}\right\}\right.$$

$$\left\{1+\frac{m^2-\tfrac{1}{2}}{2n}+\frac{3C_2}{(2n)^2}+\cdots\cdots\right\}+\sin\left\{(n+\tfrac{1}{2})\theta+\frac{m\pi}{2}-\frac{\pi}{4}\right\}$$

$$\left.\left\{-\frac{m^2-\tfrac{1}{4}}{2n}\cot\theta-\frac{3C_2'}{(2n)^2}-\cdots\right\}\right](2)$$

The values of n for which $P_n{}^m(\cos\theta)$ vanishes, will make

$$\cos\left\{(n+\tfrac{1}{2})\theta+\frac{m\pi}{2}-\frac{\pi}{4}\right\}\left\{1+\frac{m^2-\tfrac{1}{2}}{2n}+\frac{3C_2}{(2n)^2}+\cdots\right\}$$

$$+\sin\left\{(n+\tfrac{1}{2})\theta+\frac{m\pi}{2}-\frac{\pi}{4}\right\}\left\{-\frac{m^2-\tfrac{1}{4}}{2n}\cot\theta-\frac{3C_2'}{(2n)^2}\cdots\cdots\right\},$$

vanish.

Therefore, the required values of n are obtained from the equation

$$\cos\left\{(n+\tfrac{1}{2})\theta+\frac{m\pi}{2}-\frac{\pi}{4}\right\}\left\{1+\frac{m^2-\tfrac{1}{2}}{2n}+\frac{3C_2}{(2n)^2}+\cdots\right\}$$

$$+\sin\left\{(n+\tfrac{1}{2})\theta+\frac{m\pi}{2}-\frac{\pi}{4}\right\}\left\{-\frac{m^2-\tfrac{1}{4}}{2n}\cot\theta-\frac{3C'_2}{(2n)^2}-\cdots\right\}=0\cdots(3)$$

Putting $\left\{ (n+\tfrac{1}{2})\,\theta + \dfrac{m\pi}{2} - \dfrac{\pi}{4} \right\} = \phi,$

$$R = 1 + \frac{^2 - \tfrac{1}{4}}{2n} + \frac{3C_2}{(2n)^2} \;+\; \quad \cdots$$

$$S = -\frac{m^2 - \tfrac{1}{4}}{2n}\;\cot\theta - \frac{3C'_2}{(2n)^2} \;-\; \quad \cdots$$

the equation (3) reduces to

$$R\cos\phi + S\sin\phi = 0 \qquad (4)$$

Putting $R = M\cos\psi$, $S = M\sin\psi$ in (4), where $M = \sqrt{R^2 + S^2}$ and $\tan\psi = \dfrac{S}{R}$, we obtain

$$\cos(\phi - \psi) = 0.$$

Hence, $\qquad\qquad \phi = (2k+1)\,\dfrac{\pi}{2} + \psi \qquad (5)$

where k is zero or an integer.

We have

$$\tan\psi = \frac{-\dfrac{m^2-\tfrac{1}{4}}{2n}\cot\; -\dfrac{3C'_2}{(2n)^2}\; -}{1 + \dfrac{m^2-\tfrac{1}{2}}{2n} + \dfrac{3C_2}{(2n)^2}\; +}$$

$$= -\frac{m^- - \tfrac{1}{2}}{2n}\;\cot\theta + \frac{(m^2-\tfrac{1}{4})\,(m^2-\tfrac{1}{4})\cot\theta - 3\,C'_2}{(2n)^2}$$

$$+ \frac{3\,(m^2-\tfrac{1}{4})\,C_2\,\cot\theta}{(2n)^3} \;+\; \quad \cdots \quad \cdots$$

Substituting the values of ϕ and ψ in (5), we get

$$n = \frac{\pi}{2\theta}\left\{ 2k - m + \tfrac{3}{2} - \frac{\theta}{\pi} \right\} + \frac{1}{\theta}\;\tan^{-1}\left\{ -\frac{m^2-\tfrac{1}{2}}{2n}\cot\theta \right.$$

$$\left. + \frac{(m^2-\tfrac{1}{4})\,(m^2-\tfrac{1}{4})\cot\theta - 3C'_2}{(2n)^2} + \frac{3(m^2-\tfrac{1}{4})C_2\;\cot\theta}{(2n)^3} + \quad \cdots \right\} \cdots (6)$$

Putting $\dfrac{\pi}{2\theta}\left\{ 2k - m + \tfrac{3}{2} - \dfrac{\theta}{\pi} \right\} = \xi$ in (6) and expanding it by Lagrange's theorem, we obtain

$$n = \xi + \frac{1}{\theta}\left\{ -\frac{m^2-\tfrac{1}{4}}{2\xi}\;\cot\theta + \frac{(m^2-\tfrac{1}{4})\,(m^2-\tfrac{1}{4})\cot\theta - 3\,C'_2}{(2\xi)^2} \right.$$

$$\left. + \frac{3\,(m^2-\tfrac{1}{4})\,C_2\;\cot\theta}{(2\xi)^3} + \frac{(m^2-\tfrac{1}{4})^2\;\cot^3\theta}{3(2\xi)^3} + \quad \cdots \quad \cdots \right\}$$

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$$+ \frac{1}{\theta^2}\left\{ - \frac{(m^2-\frac{1}{4})^2 \cot^2\theta}{2^2\xi^2} - \frac{2\{(m^2-\frac{1}{4})(m^2-\frac{1}{4})\cot\theta - 3C'_2\}^2}{2^4\xi^4}\right.$$

(7)

This expression may be compared with the one obtained by Macdonald. Macdonald's expression can be written in the form.

$$n = -\tfrac{1}{2} + x_o + \frac{b_1}{\theta(1+x_o)} + \frac{b_2}{\theta(1+x_o)(2+x_o)} - \frac{a_1 b_1}{\theta(1+x_o)^2} + \frac{3a_1 b_1 - b_1^3}{3\theta(1+x_o)^3}$$

$$- \frac{a_2 b_1 + a_1 b_2}{\theta(1+x_o)^2(2+x_o)} + \frac{b_1}{\theta(1+x_o)(2+x_o)(3+x_o)} \quad \frac{b_1}{\theta^2(1+x_o)^3}$$

$$+ \frac{a_1 b_1^3 - a_1^3 b_1}{\theta(1+x_o)^4} + \frac{2a_1 a_2 b_1 + a_1^2 b_2 - b_1^2 b_2}{\theta(1+x_o)^3(2+x_o)} - \frac{a_2 b_2}{\theta(1+x_o)^2(2+x_o)^2}$$

$$\frac{a_1 b_2 + a_2 b_1}{\theta(1+x_o)^2(2+x_o)(3+x_o)} \cdot \frac{b_4}{\theta(1+x_o)(2+x_o)(3+x_o)(4+x_o)}$$

$$- \frac{3a_1 b_1}{\theta^2(1+x_o)^4} - \frac{2b_1 b_2}{\theta^2(1+x_o)^3(2+x_o)} + \frac{b_1 b_2}{\theta^2(1+x_o)^2(2+x_o)}, + \text{etc.},$$

where

$$x_o = \frac{\pi}{2\theta}(2k+2s-m+\tfrac{1}{2}),$$ s being the greatest integer less than m, and

k has all positive integral values including zero; and

$$\frac{1-4m^2}{2^2}\frac{\cos\left(\frac{\pi}{2}-\theta\right)}{2\sin\theta}, \qquad b_1 = \frac{1-4m^2}{2^2}\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{2\sin\theta},$$

$$a_2 = \frac{(1-4m^2)(3^2-4m^2)}{2^4(2\sin\theta)^2 2!}\cos(\pi-2\theta), \quad b_2 = \frac{(1-4m^2)(3^2-4m^2)}{2^4(2\sin\theta)^2 2!}\sin(\pi-2\theta),$$

etc.   etc.

It is easy to see that the asymptotic expansion for $\frac{d}{d\theta}P_n^m(\cos\theta)$, can be written in the form.

$$\frac{d}{d\theta}P_n^m(\cos\theta) = \frac{\Gamma(n+1)}{\Gamma(n-m+1)}\frac{1}{\sqrt{2n\pi}}\left[e^{\left\{(n+\frac{1}{2})\theta + \frac{m\pi}{2} - \frac{\pi}{4}\right\}i}\right.$$

$$\left. \sum_{s=o} \frac{A_s \Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2})n^s} + e^{-\left\{(n+\frac{1}{2})\theta + \frac{m\pi}{2} - \frac{\pi}{4}\right\}i}\sum_{s=o}^{\infty} \frac{A'_s \Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2})n^s}\right]$$ (8)

where

$$A_s = \frac{ic_s}{\sqrt{\sin\theta}}(n+\tfrac{1}{2}) + \frac{d}{d\theta}\left(\frac{c_s}{\sqrt{\sin\theta}}\right),$$

$$A'_s = - \frac{ic'_s}{\sqrt{\sin\theta}}(n+\tfrac{1}{2}) + \frac{d}{d\theta}\left(\frac{c'_s}{\sqrt{\sin\theta}}\right),$$

c_s, c'_s having the same values as before.

The roots of the equation $\dfrac{d}{d\theta}\,P_n^m(\cos\theta)=0$ can therefore be obtained in a similar way.

§ 3.

Prof. Macdonald has not attempted to obtain the values of n which make $\dfrac{d}{d\mu}\,P_n^m(\mu)$ vanish. I shall now show that Hobson's formula can also be used for the determination of the roots of this equation.

We have

$$P_n^m(\cos\theta) = \frac{2\Pi(n+m)}{\sqrt{\pi}\,\Pi(n+\tfrac{1}{2})}\left[\frac{\cos\left\{(n+\tfrac{1}{2})\theta - \tfrac{\pi}{4} + \tfrac{m\pi}{2}\right\}}{(2\sin\theta)} \right.$$
$$+ \frac{1^2-4m^2}{2(2n+3)}\frac{\cos\left\{(n+\tfrac{3}{2})\theta - \tfrac{3\pi}{4} + \tfrac{m\pi}{2}\right\}}{(2\sin\theta)^2}$$
$$+ \frac{(1^2-4m^2)(3^2-4m^2)}{2\cdot4\,(2n+3)(2n+5)}\frac{\cos\left\{(n+\tfrac{5}{2})\theta - \tfrac{5\pi}{4} + \tfrac{m\pi}{2}\right\}}{(2\sin\theta)^{\tfrac{5}{2}}} + \; \dots \left.\right] \quad (9)$$

provided $\tfrac{1}{6}\pi < \theta < \tfrac{5}{6}\pi$.

Changing n into $n-1$ in (9) we get

$$P_{n-1}^m(\cos\theta) = \frac{2\Pi(n+m-1)}{\sqrt{\pi}\,\Pi(n-\tfrac{1}{2})}\left[\frac{\cos\left\{(n-\tfrac{1}{2})\theta - \tfrac{\pi}{4} + \tfrac{m\pi}{2}\right\}}{(2\sin\theta)}\right.$$
$$+ \frac{1^2-4m^2}{2(2n+1)}\frac{\cos\left\{(n+\tfrac{1}{2})\theta - \tfrac{3\pi}{4} + \tfrac{m\pi}{2}\right\}}{(2\sin\theta)^{\tfrac{3}{2}}}$$

$$+ \frac{(1^2-4m^2)\,(3^2-4m^2)}{2\cdot4\cdot(2n+1)\,(2n+3)} \frac{\cos\left\{(n+\tfrac{3}{2})\theta - \frac{5\pi}{4} + \frac{m\pi}{2}\right\}}{(2\sin\theta)^{\frac{5}{2}}} + \ \ldots \ \Big] (10)$$

and using the relation

$$(\mu^2-1)\,\frac{d}{d\mu}\,P_n^m\,(\mu) = n\mu\ P_n^m\,(\mu) - (n+m)\ P_{n-1}^m(\mu)$$

we get

$$(\mu^2-1)\,\frac{d}{d\mu}\,P_n^m\,(\mu) = \frac{2\Pi\,(n+m)}{\sqrt{\pi}\,\Pi(n-\tfrac{1}{2})} \left[-\cos\ \frac{\left\{(n-\tfrac{1}{2})\,\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right\}}{(2\sin\theta)^{\frac{1}{2}}} \right.$$

$$\sin\ \frac{\left\{(n+\tfrac{1}{2})\,\theta - \frac{3\pi}{4} + \frac{m\pi}{2}\right\}}{(2\sin\theta)^{\frac{1}{2}}}\left\{\frac{n\mu}{n+\tfrac{1}{2}}\right\}$$

$$-\frac{\cos\ \left\{(n+\tfrac{1}{2})\,\theta - \frac{3\pi}{4} + \frac{m\pi}{2}\right\}}{(2\sin\theta)^{\frac{3}{2}}}\left\{\frac{(1^2-4m^2)}{2\cdot(2n+1)}\right\}$$

$$\sin\ \frac{\left\{n+\tfrac{3}{2})\theta - \frac{5\pi}{4} + \frac{m\pi}{2}\right\}}{(2\sin\theta)^{\frac{3}{2}}}\left\{\frac{n\mu}{n+\tfrac{1}{2}}\ \frac{1^2-4m^2}{2(2n+3)}\right\}$$

$$-\frac{\cos\ \left\{(n+\tfrac{3}{2})\,\theta - \frac{5\pi}{4} + \frac{m\pi}{2}\right\}}{(2\sin\theta)^{\frac{5}{2}}}\left\{\frac{(1^2-4m^2)\,(3^2-4m^2)}{2\cdot4\cdot(2n+1)\,(2n+3)}\right.$$

$$\left. \ldots\right] \qquad\qquad (11)$$

The large values of n for which $\dfrac{d}{d\mu}\,P_n^m(\mu)$ vanishes are given by the equation

$$\cos\left\{(n-\tfrac{1}{2})\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right\} - \mu\cos\left\{(n+\tfrac{1}{2})\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right\} = 0 \ \ldots \ (12)$$

Putting $\quad(n-\tfrac{1}{2})\,\theta - \frac{\pi}{4} + \frac{m\pi}{2} = \phi,$

the equation reduces to

$$\cos\phi - \mu\cos\,(\phi+\theta) = 0,$$

or

$$(1-\mu^2)\,\cos\phi + \mu\ \sqrt{(1-\mu^2)}\ \sin\phi = 0. \qquad\qquad (13)$$

Again putting

$$(1-\mu^2)=r\cos\beta, \quad \mu\sqrt{1-\mu^2}=r\sin\beta,$$

where $r=\sqrt{1-\mu^2}$ and $\tan\beta=\dfrac{\mu}{\sqrt{(1-\mu^2)}}$

the equation becomes

$$\cos(\phi-\beta)=o.$$

Hence

$$\phi=(2k+1)\frac{\pi}{2}+\beta, \qquad \dots \qquad \dots \qquad \dots (14)$$

where k is zero or an integer.

Substituting the values of ϕ and β we have

$$n-\tfrac{1}{2}=\frac{\pi}{2\theta}(2k-m+\tfrac{3}{2})+\frac{1}{\theta}\tan^{-1}\frac{\mu}{\sqrt{1-\mu^2}}$$

$$=\frac{\pi}{2\theta}(2k-m+\tfrac{3}{2})-1.$$

Put $n-\tfrac{1}{2}=x$, $\dfrac{\pi}{2\theta}(2k-m+\tfrac{3}{2})=x_o;$

then for all positive values of n which make $\dfrac{d}{d\mu}P^m_n(\mu)$ vanish, we

have $(x-x_o)\theta=\psi$, where ψ is determined by the equation

$$\tan\psi=\frac{L}{M},$$

where $L=\dfrac{\mu}{1+\dfrac{1}{1+2x}}\cos\left(\dfrac{\pi}{2}-\theta\right)+\dfrac{1-4m^2}{2^3(1+x)}\dfrac{\sin\left(\dfrac{\pi}{2}-\theta\right)}{2\sin\theta}$

$$+\frac{(1-4m^2)\mu}{2^3(2+x)\left(1+\dfrac{1}{1+2x}\right)}\frac{\cos(\pi-2\theta)}{2\sin\theta}+$$

and $M=1-\dfrac{\mu}{1+\dfrac{1}{1+2x}}\sin\left(\dfrac{\pi}{2}-\theta\right)+\dfrac{1-4m^2}{2^3(1+x)}\dfrac{\cos\left(\dfrac{\pi}{2}-\theta\right)}{2\sin\theta}$

$$\frac{(1-4m^2)\mu}{2^3(2+x)\left(1+\dfrac{1}{1+2x}\right)}\frac{\sin(\pi-2\theta)}{2\sin\theta}+ \qquad \dots (15)$$

Now putting

$$\frac{\mu}{2}\cos\left(\frac{\pi}{2}-\theta\right)=b_1,\quad \frac{1-4m^2}{2^2}\;\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{2\sin\theta}=b_2,$$

$$\frac{(1-4m^2)\mu}{2^3}\;\frac{\cos\,(\pi-2\theta)}{2\sin\theta}=b_3,\quad\ldots$$

$$-\frac{\mu}{2}\sin\left(\frac{\pi}{2}-\theta\right)=a_1,\quad \frac{1-4m^2}{2^2}\;\frac{\cos\left(\frac{\pi}{2}-\theta\right)}{2\sin\theta}=a_2,$$

$$-\frac{(1-4m^2)\mu}{2^3}\;\frac{\sin\,(\pi-2\theta)}{2\sin\theta}=a_3,\quad\ldots$$

we get

$$L=\frac{b_1(1+2x)}{1+x}+\frac{b_2}{1+x}+\frac{b_3(1+2x)}{(1+x)(2+x)}+\frac{b_4}{(1+x)(2+x)}$$

$$+\frac{b_5(1+2x)}{(1+x)(2+x)(3+x)}+\frac{b_6}{(1+x)(2+x)(3+x)}+\ \ldots$$

$$M=1+\frac{a_1(1+2x)}{1+x}+\frac{a_2}{1+x}+\frac{a_3(1+2x)}{(1+x)(2+x)}+\frac{a_4}{(1+x)(2+x)}$$

$$+\frac{a_5(1+2x)}{(1+x)(2+x)(3+x)}+\frac{a_6}{(1+x)(2+x)(3+x)}+\ \ldots$$

The expression for $\tan\psi$ can therefore be written in the form

$$\tan\psi=\frac{b_1(1+2x)}{1+a_1+(1+2a_1)x}+\frac{b_2}{1+a_1+(1+2a_1)x}$$

$$+\frac{b_3(1+2x)}{\{1+a_1+(1+2a_1)x\}(2+x)}-\frac{a_2b_1(1+2x)}{\{1+a_1+(1+2a_1)x\}^2}$$

$$-\frac{a_3b_1(1+2x)^2}{\{1+a_1+(1+2a_1)x\}^2(2+x)}+\frac{b_4}{\{1+a_1+(1+2a_1)x\}(2+x)}$$

$$+\frac{b_5(1+2x)}{\{1+a_1+(1+2a_1)x\}(2+x)(3+x)}-\frac{a_4b_1(1+2x)}{\{1+a_1+(1+2a_1)x\}^2(2+x)}$$

$$-\frac{a_5b_1(1+2x)^2}{\{1+a_1+(1+2a_1)x\}^2(2+x)(3+x)}-\frac{(a_2b_3+a_3b_2)(1+2x)}{\{1+a_1+(1+2a_1)x\}^2(2+x)}$$

$$-\frac{a_2b_2}{\{1+a_1+(1+2a_1)x\}^2}-\frac{a_3b_3(1+2x)^2}{\{1+a_1+(1+2a_1)x\}^2(2+x)^2}$$

$$+\frac{a_2{}^2b_1(1+2x)}{\{1+a_1+(1+2a_1)x\}^3}+\frac{a_3{}^2b_1(1+2x)^3}{\{1+a_1+(1+2a_1)x\}^3(2+x)^2}$$

$$+\frac{2a_2a_3b_1(1+2x)^2}{\{1+a_1+(1+2a_1)x\}^3(2+x)}+\ldots\qquad\ldots\qquad\ldots\quad(16)$$

Putting

$$\frac{b_1(1+2c)}{1+a_1+(1+2a_1)x} = a_0$$

$$\frac{b_2}{1+a_1+(1+2a_1)x} + \cdots - \frac{a_3 b_1(1+2c)^2}{\{1+a_1+(1+2a_1)x\}^2(2+x)} = a_1$$

...   .

...   .

the equation (16) assumes the form

$$\tan (x-x_0) \theta = a_0 + a_1 + a_2 + a_3 + \text{ etc.}$$

Whence

$$x = x_0 + \frac{1}{\theta} \tan^{-1} (a_0 + a_1 + a_2 + a_3 + \cdots), \qquad \ldots (17)$$

Now $\tan^{-1} (a_0 + a_1 + a_2 + a_3 + \ldots)$ can be regarded as a function of x, and consequently we can expand this by Lagrange's theorem and we have

$$x = x_0 + \frac{1}{\theta} \left\{ a_0 + a_1 + a_2 + a_3 - \frac{a_0^3}{3} - \frac{a_1^3}{3} - a_0^2 a_1 - a_0^2 a_2 - a_0^2 a_3 - a_1^2 a_2 - a_0 a_1^2 \right.$$

$$\left. -2a_0 a_1 a_2 \right\} + \frac{1}{\theta^2} \left\{ a_0 a_0' + a_0 a_2' + a_0 a_1' + a_1 a_1' + a_0' a_1 + a_0' a_2 \right\} + \ldots (18)$$

Here x_0 is written for x in $a_0, a_1, a_2 \ldots \ldots \ldots$ and $a_0', a_1' \ldots \ldots \ldots$ mean $\frac{da_0}{dx_0}, \frac{da_1}{dx_0}, \ldots \ldots \ldots$ and terms of the order $\frac{1}{(1+x_0)^4}, \frac{1}{(1+x_0)^5}$, etc., are neglected.

Substituting for $a_0, a_1 \ldots \ldots \ldots$ their values, we have

$$x = x_0 + \frac{1}{\theta} \left\{ \frac{b_1(1+2x_0)}{1+a_1+(1+2a_1)x_0} + \frac{b_2}{\{1+a_1+(1+2a_1)x_0\}} + \ldots \right\}$$

$$+ \frac{1}{\theta^2} \left[\frac{b_1(1+2x_0)}{1+a_1+(1+2a_1)x_0} \left\{ \frac{2b_1}{1+a_1+(1+2a_1)x_0} \right. \right.$$

$$\left. \left. - \frac{b_1(1+2a_1)(1+2x_0)}{\{1+a_1+(1+2a_1)x_0\}^2} \right\} + \ldots \right] + \ldots \ldots \ldots \qquad \ldots (19)$$

The negative values of n for which $\frac{d}{d\mu} P_n^m(\mu)$ vanishes are obtained from the above by changing the sign of the right-hand side by the relation

$$P_n^m (\mu) = P_{-n-1}^m(\mu).$$

The numerical values of the first few roots of the equation $\frac{d}{d\mu} P_n^m(\mu) = 0$ are given in Tables I, II, and III.

TABLE I.

$$\vartheta = \frac{\pi}{4}, m=0$$

k	x_0	a_0	a_1	a_2	$a'_0 a_0$	$a'_0 a_1$	$a_0 a_1'$	n
1	7	·8823	·0250	·0033	·0122	·0003	−·0022	8·3:
2	11	·9200	·0178	·0012	·0058	·0001	−·0011	12·3:
3	15	·9393	·0138	·0009	·0034	·0000	−·0006	16·3:
4	19	·9512	·0113	·0006	·0022	·0000	−·0003	20·3:
5	23	·9591	·0096	·0004	·0015	·0000	−·0001	24·3:

TABLE II.

$$\theta = \frac{\pi}{4}, m=1$$

k	x_0	a_0	a_1	a_2	$a'_0 a_0$	$a'_0 a_1$	$a_0 a_1'$	n
1	5	·8461	−·0923	−·0120	·0200	−·0022	0100	6·3:
2	9	·9047	−·0351	−·0045	·0082	−·0005	·0045	10·3:
3	13	·9344	−·0469	−·0023	·0044	−·0002	·0024	14·3:
4	17	·9459	−·0375	−·0013	·0027	−·0001	·0013	18·3:
5	21	·9555	−·0313.	−·0009	·0018	−·0000	·0007	22·3:

TABLE III.

$$\theta = \frac{\pi}{4}, m=2,$$

k	x_0	a_0	a_1	a_2	$a'_0 a_0$	$a'_0 a_1$	$a_0 a_1'$	n
1	3	·7777	−·6388	−·0234	·0384	−·0709	·0894	3·7:
2	7	·8823	−·376;	−·0086	·0122	−·0052	·0333	8·1:
3	11	·9200	−·2681	−·0044	·0058	−·0017	·0175	12·2:
4	15	·9393	−·2084	−·0023	·0034	−·0007	·0107	16·2:
5	19	·9512	−·1705	−·0014	·0022	−·0004	·0065	20·2:

On Sound-Waves due to Prescribed Vibrations on a Spherical Surface in the Presence of a Rigid and Fixed Spherical Obstacle.

By

Sudhansukumar Banerji.

In this paper I propose to give the complete solution of the problem of aerial disturbance due to any prescribed vibration on the surface of a sphere in the presence of a rigid and fixed spherical obstacle. This problem remained unsolved up to this time, although, so early as 1868, Sir George Stokes[*] solved the problem of the propagation of waves outwards from a spherical surface in an unlimited medium, in 1862, Clebsch[†] investigated the scattering of sound-waves due to a simple point-source placed at a finite distance by a spherical surface, and in 1872, Lord Rayleigh[‡] solved a similar problem of disturbance produced by a spherical surface on plane waves of sound. All the results given in this paper may therefore be regarded as new.

The method adopted may be called the method of successive reflections and has some analogy to the method of images used in solving the problem of the motion of two spheres in an infinite liquid.

§ 1.

For the sake of definiteness, I shall first consider the case when the two spheres are external to each other and the prescribed vibration given to one of them is symmetrical about the line joining the two centres.

Let A and B be the two spheres and a and b their radii.

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[*] Stokes, On the communications of vibrations form a vibrating body to a surrounding gas, *Phil. Trans. 1968*. [*Math. and Phys. Papers, Vol. IV, p. 299.*]

[†] Clebsch, Über die Reflexion an einer Kugelfläche, *J. f. Math. Bd. 61, p. 195,* (1863).

[‡] Lord Rayleigh, Investigation of the disturbance produced by a spherical obstacle on the waves of sound, *Proc. Lond. Math. Soc., Vol. IV, p. 253, (1872).* [*Theory of Sound, Art. 334.*]

The polar co-ordinates of any external point P are (r, θ, ϕ) referred of the centre of the sphere A, and (r', θ', ϕ) referred to the centre of the sphere B, θ and θ' being measured in opposite senses from the line A B.

Since the prescribed vibration on the surface of the sphere A (say) is symmetrical about the line joining the two centres, it must be expressible by a series of the type

$$\sum_{n=o}^{n=m} A_n \, P_n \, (\cos \theta) \, e^{ikct} \quad ,$$

where the A_n's are known constants, and m may be finite or infinite.

§ 2.

If we ignore the presence of the spherical obstacle B in the medium, then there is nothing to obstruct the wave motion started by the sphere A and the problem reduces to the one whose complete solution was obtained by Sir George Stokes* in 1868.

If ψ_0 denote the velocity potential of this unobstructed wave motion then ψ_0 is given by

$$\psi_0 = \sum_{n=o}^{n=m} A_n \, \frac{\dfrac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}}{\dfrac{d}{da}\left[\dfrac{K_{n+\frac{1}{2}}(ika)}{\sqrt{a}}\right]} \, P_n \, (\cos \theta) \, e^{ikct}$$

I shall write this in the form

$$\psi_0 = \sum_{n=o}^{n=m} A_n^{(o)} \, \frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} \, P_n \, (\cos \theta) \, e^{ikct} \quad .$$

§ 3.

I shall now consider the effect of the presence of the spherical obstacle B in the medium.

For this purpose, I shall first regard the dimensions of the vibrating sphere to be very small in comparison with the wave-length, that is, I shall practically regard it as a multiple point-source of sound of the m th and lower degrees of complexity and calculate the scattering of the waves produced by the presence of the spherical obstacle B.

The velocity potential of the wave system incident on the surface of the sphere B is of course given by the above expression for ψ_0.

---

* Stokes, loc. cit.

Let the velocity potential of the scattered wave system regarded as radiating outwards from the centre of the spherical obstacle B be denoted by ψ_1.

Then the function ψ_1 has to satisfy the following conditions :—

(i) it must satisfy $\dfrac{d^2\psi}{dt^2} = c^2 \nabla^2 \psi$;

(ii) it must satisfy the condition $\dfrac{d\psi_0}{dr'} + \dfrac{d\psi_1}{dr'} = 0$,

on the surface of the sphere B, i.e., on $r' = b$;

(iii) together with its first and second derivatives it must be finite and continuous everywhere in the medium ;

(iv) it must vanish as $\dfrac{e^{-ikr'}}{r'}$, when r' tends to infinity.

We can therefore assume ψ_1 to be given by the series

$$\psi_1 = \sum_{p=0}^{p=\infty} A_p^{(1)} \frac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}} P_p(\cos \theta') e^{ikct} .$$

This expression satisfies the conditions (i) and (iii). It also satisfies the condition (iv) for the reason that when r is large

$$K_p(ir) = \sqrt{\frac{\pi}{2r}} e^{-i\left(r+\frac{\pi}{4}\right)}$$

approximately.*

§ 4.

The unknown constants $A_p^{(1)}$'s in the expression for ψ_1 have to be determined by means of the condition (ii).

If d is the distance between the centres of the two spheres, then

$$r^2 = r'^2 + d^2 - 2r'd \cos \theta'.$$

If we write $\cos \theta = \mu$ and $\cos \theta' = \mu'$, we can always expand the function $\dfrac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n(\mu)$ as an infinite series of functions of the type $\dfrac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\mu')$, provided $r' < d$.

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* Cf., Nielsen, *Handbuch der Theorie der Cylinderfunktionen*, p. 154.

We can in fact write

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\,P_n(\mu) = \sum_{p=0}^{p=\infty} A_{n,p}\,\frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}\,P_p(\mu'),$$

and it is easy to prove that

$$A_{n,p} = \frac{(2p+1)i^p}{2}\cdot \operatorname*{Lim.}_{a=0}\int_{-1}^{1} P_n(z)P_p(z)\phi(a,d,z)dz$$

where

$$\phi(a,d,z) = \sqrt{\frac{2\pi}{kd}}\,\sum_{m=0}^{m=\infty} e^{-m^2 a}\,(m+\tfrac{1}{2})\,K_{m+\frac{1}{2}}(ikd)\,P_m(z).$$

[The proof can be briefly stated as follows :—

We have

$$\frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\,P_n(\mu) = i^{-n}\,\sqrt{\frac{2k}{\pi}}\,P_n\!\left(\frac{d}{id(k.c)}\right)\frac{\sin kr}{kr},$$

and

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\,P_n(\mu) = (-1)^n\,\sqrt{\frac{\pi k}{2i}}\,P_n\!\left(\frac{d}{id(k.c)}\right)\frac{e^{-ikr}}{kr},$$

the x—axis being the line AB.

Also

$$\frac{e^{-ikr}}{r} = \frac{2\sqrt{i}}{\sqrt{dr'}}\,\sum_{m=0}^{m=\infty} i^m(m+\tfrac{1}{4})\,K_{m+\frac{1}{2}}(ikd)\,J_{m+\frac{1}{2}}(kr')\,P_m(\mu'),$$

if $r' < d$.

Therefore

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\,P_n(\mu) = \sqrt{\frac{\pi}{2kd}}\,\sum_{m=0}^{m=\infty} i^m(2m+1)\,P_n\!\left(\frac{d}{id(k.c')}\right)$$

$$\times K_{m+\frac{1}{2}}(ikd)\,\frac{J_{m+\frac{1}{2}}(kr')}{\sqrt{r'}}\,P_m(\mu'),$$

where the x'—axis is the line BA and has the sense BA, so that

$$d - \frac{d}{l.c'}$$

Thus

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\,P_n(\mu) = \frac{2}{\sqrt{d}}\sum_{m=o}^{m=\infty}(m+\tfrac{1}{2})\,K_{m+\frac{1}{2}}(ikd)$$

$$\times P_n\!\left(\frac{d}{id(kx')}\right)P_m\!\left(\frac{d}{id(kx')}\right)\frac{\sin kr'}{kr'}$$

$$= \frac{2}{\sqrt{d}}\sum_{m=o}^{m=\infty}(m+\tfrac{1}{2})\,K_{m+\frac{1}{2}}(ikd)\Sigma C_{m,n,p}\,P_p\!\left(\frac{d}{id(kx')}\right)\frac{\sin kr'}{kr'},$$

where $\overset{*}{C}_{m,n,p} = \dfrac{2p+1}{2}\displaystyle\int_{-1}^{\cdot}P_m(z)P_n(z)P_p(z)\,dz.$

$$= \frac{2p+1}{m+n+p+1}\,\frac{A\!\left(\dfrac{m+p-n}{2}\right)A\!\left(\dfrac{n+m-p}{2}\right)A\!\left(\dfrac{n+p-m}{2}\right)}{A\!\left(\dfrac{n+m+p}{2}\right)},$$

$$A(m) = \frac{1.3.5\ldots 2m-1}{1.2.3\ldots\ m}\quad\text{and }p\text{ takes the values}$$

$$n-m,\ n-m+2,\ n-m+4,\ \ldots\ n+m,\ \text{if } n>m.$$

$$m-n,\ m-n+2,\ m-n+4,\ \ldots\ m+n,\ \text{if } m>n.$$

Therefore

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\,P_n(\mu) = \sqrt{\frac{2\pi}{kd}}\sum_{m=o}^{m=\infty}(m+\tfrac{1}{2})\,K_{m+\frac{1}{2}}(ikd)$$

$$\times\,\Sigma C_{m,n,p}\,i^p\,\frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}\,P_p(\mu') = \sqrt{\frac{2\pi}{kd}}\sum_{p=o}^{p=\infty}\frac{2p+1}{2}\,i^p$$

$$\times\,\frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}\,P_p(\mu').\underset{a=0}{\text{Lim.}}\int_{-1}^{1}P_n(z)P_p(z)\left\{\sum_{m=o}^{m=\infty}e^{-m^2 a}(m+\tfrac{1}{2})\right.$$

$$\left.K_{m+\frac{1}{2}}(ikd)\,P_m(z)\right\}dz\,\Big].$$

* Adams, *Proc. R. S.*, (1878), Collected Scientific Papers, Vol. 1, pp. 487-496.

Near the surface of the sphere B we have $r' < d$, so that by using the above expansion and by substitution in the boundary condition

$$\frac{d\psi_0}{dr} + \frac{d\psi_1}{dr'} = o, \text{ for } r' = b,$$

and on equating the co-efficient of $P_p(\mu')$ to zero, we at once obtain

$$A_p^{(1)} = -\left\{ \frac{d}{db} \frac{J_{p+\frac{1}{2}}(kb)}{\sqrt{b}} \Big/ \frac{d}{db} \frac{K_{p+\frac{1}{2}}(ikb)}{\sqrt{b}} \right\} \frac{2p+1}{2} i^p$$

$$\times \lim_{a=0} \int_{-1} P_p(z)\phi(a, d, z) \times \left\{ \sum_{n=0}^{n=m} A_n^{(o)} P_n(z) \right\} dz,$$

and therefore

$$\psi_1 = -\sum_{p=0}^{p=\infty} \left\{ \frac{d}{db} \frac{J_{p+\frac{1}{2}}(hb)}{\sqrt{b}} \Big/ \frac{d}{db} \frac{K_{p+\frac{1}{2}}(ikb)}{\sqrt{b}} \right\} \frac{(2p+1)}{} i^p$$

$$\times \frac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}} P_p(\cos\theta') e^{ikct} \times \lim_{a=0} \left[\int_{-1} P_p(z)\phi(a, d, z) \right.$$

$$\times \left\{ \sum_{n=0}^{n=m} A_n^{(o)} P_n(z) \right\} dz \left. \right].$$

§ 5.

Now if we suppose that the vibrating sphere has a finite size then the scattered waves whose velocity potential we have determined in the last article will be incident on the vibrating sphere and will be again scattered. Let ψ_2 be the velocity potential of these last mentioned scattered waves regarded as radiating outwards from the centre of the sphere A. In determining the value of ψ_2, we ignore the presence of the sphere B.

Like ψ_1, ψ_2 has also to satisfy the conditions (i), (iii) and (iv) stated in § 3. The condition (ii) however for this case becomes

$$\frac{d\psi_1}{dr} + \frac{d\psi_2}{dr} = 0 \text{ for } r = a.$$

We can therefore assume ψ_2 to be given by the series

$$\psi_2 = \sum_{p=0}^{p=\infty} A_p^{(2)} \frac{K_{p+\frac{1}{2}}(ikr)}{\sqrt{r}} P_p(\cos\theta) e^{ikct}.$$

Proceeding as in § 4, we find that

$$A_p^{(2)} = -\left\{ \frac{d}{da} \frac{J_{p+\frac{1}{2}}(ka)}{\sqrt{a}} \bigg/ \frac{K_{p+\frac{1}{2}}(ika)}{\sqrt{a}} \right\} \frac{(2p+1)}{2} i^p$$

$$\underset{a=0}{\text{Lim.}} \int_{-1}^{1} P_p(z) \phi(a, d, z) \times \left\{ \sum_{n=0}^{n=\infty} A_n^{(1)} P_n(z) \right\} dz,$$

and therefore

$$\psi_2 = -\sum_{p=0}^{p=\infty} \left\{ \frac{d}{da} \frac{J_{p+\frac{1}{2}}(ka)}{\sqrt{a}} \bigg/ \frac{d}{da} \frac{K_{p+\frac{1}{2}}(ika)}{\sqrt{a}} \right\} \frac{(2p+1)}{2} i^p$$

$$\times \frac{K_{p+\frac{1}{2}}(ikr)}{\sqrt{r}} P_p(\cos\theta) e^{ikct}$$

$$\underset{a=0}{\text{Lim}} \left[\int_{-1}^{1} P_p(\ ) \phi(a, d, z) \left\{ \sum_{n=0}^{n=\infty} A_n^{(1)} P_n(z) \right\} dz \right].$$

§ 6.

We suppose that the scattered wave system determined by ψ_2 is again incident on the surface of the sphere B and is again scattered and that in this way an infinite number of reflections occur alternately on the two surfaces. The conditions satisfied by the velocity potentials of all these reflected wave systems are similar to those satisfied by ψ_1 and ψ_2 and all these potentials are determined in a similar manner.

The complete solution of the problem is therefore given by the following series

$$\psi = \psi_0 + \psi_1 + \psi_2 + \ldots \ldots + \psi_{2l} + \psi_{2l+1} \ldots \ldots .$$

where

$$\psi_{2l} = \sum_{p=0}^{p=\infty} A_p^{(2l)} \frac{K_{p+\frac{1}{2}}(ikr)}{\sqrt{r}} P_p(\cos\theta), e^{ikct}$$

$$\psi_{2l+1} = \sum_{p=0}^{p=\infty} A_p^{(2l+1)} \frac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}} P_p(\cos\theta'), e^{ikrt}$$

$$A_p^{(2l)} = -\left\{ \frac{d}{da} \frac{J_{p+\frac{1}{2}}(ka)}{\sqrt{a}} \bigg/ \frac{d}{da} \frac{K_{p+\frac{1}{2}}(ika)}{\sqrt{a}} \right\} \frac{(2p+1)}{2} i^p$$

$$\underset{a=o}{\text{Lim}} \int_{-1}^{1} P_p(z) \phi(a, d, z) \times \left\{ \sum_{n=o}^{n=\infty} A_n^{(2l-1)} P_n(z) \right\} dz,$$

and $A_p = - \left\{ \dfrac{d}{db} \dfrac{J_{p+\frac{1}{2}}(kb)}{\sqrt{b}} \middle/ \dfrac{d}{db} \dfrac{K_{p+\frac{1}{2}}(ikb)}{\sqrt{b}} \right. \dfrac{(2p+1)}{2}$

$$\operatorname*{Lim}_{a=0} \int_{-1}^{1} P_p(z)\, \phi(a, d, z) \times \sum_{n=0}^{n=\infty} A_n^{(2l)} P_n(z)\ dz.$$

7.

I now take up the case when one sphere is inside the other and a vibration symmetrical about the line joining the two centres is prescribed either to the internal or the external sphere. I need only consider the case of two eccentric spheres, the case of two concentric spheres being very simple and the solution easily obtained by a direct method.*

The solution for this case also is easily obtained by adopting the method of successive reflections and proceeding in almost the same manner as in the case of two external spheres. The only difference in this case will be that Bessel's functions both of the first and the second kind will enter into the solution and that we shall have to use some other theorems similar to that enunciated in § 4.

For instance, suppose that the sphere A is inside the sphere B, and that we are required to determine the disturbance produced in the space inside the two spherical surfaces, when the prescribed vibration, expressible by the series

$$\sum^{n=m} A_n P_n(\cos\theta)\, e^{ikct} \qquad .$$

is given to the inner sphere.

If ψ_0, ψ_1, ψ_2, etc., denote, as in the previous case, the velocity potentials of the initial unobstructed wave motion and the successive wave systems reflected alternately from the external and the internal surfaces, then

$$\psi_e = \sum_{n=0}^{n=m} A_r \dfrac{K_{n+\frac{1}{2}}(ikr)}{\dfrac{d}{da}\left[\dfrac{K_{n+\frac{1}{2}}(ika)}{\sqrt{a}}\right]} P_n(\cos\theta)\, e^{ikct}$$

$$= \sum_{n=0}^{n=m} A_n^{(0)} \dfrac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n(\cos\theta)\, e^{ikct},$$

* Lord Rayleigh, *Theory of Sound*, Art 333. Chree, *Messenger of Mathematics*, Vol. XV, p. 20 (1886).

and if we bear in mind that in determining the velocity potential of any reflected wave-system from any surface we ignore the presence of the other surface, we can assume the following forms for ψ_1, ψ_2, ψ_3, etc. :—

$$\psi_1 = \sum_{p=0}^{p=\infty} A_p(1) \ \frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\cos \theta') \ e^{ikct},$$

$$\psi_2 = \sum_{p=0}^{p=\infty} A_p(2) \ \frac{K_{p+\frac{1}{2}}(ikr)}{\sqrt{r}} P_p(\cos \theta) \ e^{ikct},$$

$$\psi_3 = \sum_{p=0}^{p=\infty} A_p(3) \ \frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\cos \theta') \ e^{ikct},$$

etc.

The unknown constants in each of these functions are easily determined from the boundary conditions which these functions satisfy, viz.,

$$\frac{d\psi_0}{dr'} + \frac{d\psi_1}{dr'} = 0 \text{ for } r'=b, \frac{d\psi_1}{dr} + \frac{d\psi_2}{dr} = 0 \text{ for } r=a, \text{ etc.},$$

by proceeding in a similar manner as in § 4, and by using the following two theorems :—*

(a) If $r^2 = r'^2 + d^2 - 2r'd\mu'$, then the function $\dfrac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n(\mu)$ can always be expanded as an infinite series of functions of the type $\dfrac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}} P_p(\mu')$, provided $r' > d$.

We can write this theorem in the form

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n(\mu) = \sum_{p=0}^{p=\infty} B_{n,p} \frac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}} P_p(\mu'),$$

---

* Both the theorems are easily proved by proceeding as in the theorem in § 4, if we notice that $\dfrac{e^{-ikr}}{r} = \dfrac{2}{\sqrt{dr'}} \sqrt{i} \sum_{n=0}^{n=\infty} i^n (n+\frac{1}{2}) J_{n+\frac{1}{2}}(kd) K_{n+\frac{1}{2}}(ikr') P_n(\mu'),$

$$(r' > d)$$

$$\frac{\sin kr}{r} = \frac{\pi}{\sqrt{dr'}} \sum_{n=0}^{n=\infty} (n+\frac{1}{2}) J_{n+\frac{1}{2}}(kd) J_{n+\frac{1}{2}}(kr') P_n(\mu')$$

$$\text{(for all values of } r'\text{),}$$

and

$$e^{-ikds} = \sqrt{\frac{\pi}{2kd}} \sum_{n=0}^{n=\infty} i^{3n} (2n+1) J_{n+\frac{1}{2}}(kd) P_n(z).$$

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and we easily find that

$$B_{n,p} = (2p+1)\,(-1)^p \int_{-1}^{1} P_n(z)P_p(z)e^{-ikdz}\;dz.$$

(b) Similarly, the function $\dfrac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\,P_n\,(\mu)$ can always be expanded as an infinite series of functions of the type $\dfrac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}\,P_p\,(\mu')$.

Expressing this theorem in the form

$$\frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\,P_n(\mu) = \sum_{p=o}^{p=\infty} C_{n,p}\,\frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}\,P_p\,(\mu')$$

we easily find that

$$C_{n,p} = (2p+1)\,i^{p+n}\int_{-1}^{1} P_p(z)P_n(z)e^{-ikdz}\;dz.$$

Using these two theorems, we find at once that

$$A_p^{(1)} = -\left\{\frac{d}{db}\,\frac{K_{p+\frac{1}{2}}(ikb)}{\sqrt{b}}\;\Big/\;\frac{d}{db}\,\frac{J_{p+\frac{1}{2}}(kb)}{\sqrt{b}}\right\}(2p+1)\,(-1)^p$$

$$\times \int_{-1}^{1}P_p(z)e^{-ikdz}\times\left\{\sum_{n=o}^{n=m}A_n^{(o)}\,P_n(z)\right\}dz,$$

$$A_p^{(2)} = -\left\{\frac{d}{da}\,\frac{J_{p+\frac{1}{2}}(ka)}{\sqrt{a}}\;\Big/\;\frac{d}{da}\,\frac{K_{p+\frac{1}{2}}(ika)}{\sqrt{a}}\right\}(2p+1)\,i^p$$

$$\times \int_{-1}^{1}P_p(z)e^{-idkz}\times\left\{\sum_{n=o}^{n=\infty}i^n\,A_n^{(1)}\,P_n(z)\right\}dz,$$

etc.

§ 8.

In the foregoing articles I have restricted myself to the case when the prescribed vibration is symmetrical about the line joining the centres. When it is not symmetrical about the line joining the centres, the method adopted for the symmetrical case can easily be extended to this case with the help of three theorems (involving tesseral harmonics) similar to the three theorems which I have given for the symmetrical case.

On the Vibrations of a Membrane Bounded by Two Non-Concentric Circles.

BY

Sudhansukumar Banerji.

In this paper I propose to give the complete solution of the problem of the vibrations of a membrane bounded by two circles which are not concentric. This problem remained unsolved up to this time although the corresponding problem for the case in which the circles are concentric is well-known to be easy of solution.

The method adopted is one of continued approximations and is analogous to the method used in two previous papers published by me in the Bulletin of the Calcutta Mathematical Society.*

§ 1.

Suppose that a point P has the coordinates (r,θ) and (r',θ') referrred to two points A and B, θ and θ' being measured in opposite senses from the line AB, and let $AB=\delta$ so that

$$r^2 = r'^2 + \delta^2 - 2r'\delta \cos \theta'.$$

Then the following theorems† hold true :—

(I) $(-1)^n J_n (r) \cos n\theta = \sum_{p=-\infty}^{p=\infty} (-1)^p J_p (\delta) J_{n-p} (r') \cos (n-p)\theta'.$

(II) $(-1)^n D_n (r) \cos n\theta = \sum_{p=-\infty}^{p=\infty} (-1)^p J_p(\delta) D_{n-p} (r') \cos (n-p) \theta'.$

$$\text{(if } r' > \delta),$$

or $\sum_{p=-\infty}^{p=\infty} (-1)^p D_p (\delta) J_{n-p} (r') \cos (n-p) \theta'$

$$\text{(if } r' < \delta),$$

where $D_n (r) = r^n \left(-\dfrac{d}{rdr} \right)^n D_o(r)$, $D_o(r) = \dfrac{2}{\pi} \int^{\infty} e^{-ir\cosh u} du.$

* See Vols. 4 and 5.
† See Graf and Gubler, *Einleitung in die Theorie der Bessel' schen Funktionen*, Ch. X, Art 5, pp. 81-84, also Prof. Graf's paper in *Math. Ann.*, Vol. 43.

Similar theorems will also hold true when the cosine is replaced by sine.

§ 2.

Let the membrane be bounded by two non-concentric circles of radii a and b.

Let, as before, $(r,\ \theta)$ be the polar coordinates of a point P inside the two circular boundaries referred to the centre A of the outer circle. Let $(r',\ \theta')$ be the polar coordinates of the same point referred to B, the centre of the inner circle, θ and θ' being measured in opposite senses from the line AB.

If w denote the normal displacement of a point P at any time t, it must satisfy the following conditions:—

(1)
$$\frac{d^2 w}{dt^2} = c^2\left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2}\right);$$

(2) it must vanish both on the inner and the outer boundaries;

(3) its first and second differential coefficients must be finite and continuous everywhere inside the two circular boundaries.

Let us first assume for w the following expression:—

$$w = w_0 = J_n\ (kr)\ \cos n\theta\ e^{ikct}.$$

This satisfies the differential equation (1), and will also satisfy the condition on the outer boundary, provided k is a root of the equation

$$J_n\ (ka)\ = 0. \tag{1}$$

But this will not satisfy the condition on the inner boundary.

The value of this function on the inner boundary is given by

$$w_0 = (-1)^n \sum_{p=-\infty}^{p=\infty} (-1)^p J_p\ (k\delta)\ J_{n-p}\ (kr')\ \cos (n-p)\theta'\ e^{ikct},$$

when $r' = b$.

(Theorem 1.)

To satisfy the condition on the inner boundary take a second function w_1 given by

$$w_1 = \sum_{p=-\infty}^{p=\infty} \overset{(1)}{A_{n-p}}\ D_{n-p}\ (kr')\ \cos (n-p)\theta'.\ e^{ikct}$$

This will make $w_0 + w_1$ vanish on the inner boundary if we take $A_{n-p}^{(1)}$ to be given by the expression

$$(-1)^{n+p} J_p(k\delta) J_{n-p}(kb) = -A_{n-p}^{(1)} D_{n-p}(kb),$$

i.e.,

$$A_{n-p}^{(1)} = (-)^{n+p+1} J_p(k\delta) \frac{J_{n-p}(kb)}{D_{n-p}(kb)}.$$

Since near the outer boundary $r > \delta$, we have near the outer boundary

$$w_1 = \sum_{n=-\infty}^{n=\infty} A_n^{(1)} D_n(kr') \cos n\theta' . e^{ikct}$$

$$= \sum_{n=-\infty}^{n=\infty} A_n^{(1)} (-1)^n \sum_{p=-\infty}^{p=\infty} (-1)^p J_p(k\delta) D_{n-p}(kr)$$

$$\cos(n-p)\theta . e^{ikct}$$

(Theorem II.)

To obtain the altered period correct to this order of approximation, we multiply $w_0 + w_1$ by $\cos n\theta$ and integrate with respect to θ from 0 to 2π. Hence to this order of approximation the periods are given by k which are the roots of the equation

$$\int (w_0 + w_1) \cos n\theta \, d\theta = 0,$$

that is to say, the equation

$$J_n(ka) + J_s(k\delta) D_n(ka) \sum_{s=-\infty}^{s=\infty} (-1)^s A_s^{(1)} = 0.$$

(2)

Now $w_0 + w_1$ will no longer satisfy the condition on the outer boundary. To satisfy the condition on the outer boundary, introduce a third function w_2 given by

$$w_2 = \sum_{s=-\infty}^{s=\infty} A_s^{(2)} J_s(kr) \cos s\theta . e^{ikct}.$$

So that if we write $n-p=s$, we see that $w_a+w_1+w_a$ will vanish on the outer boundary if $A_a^{(2)}$ is given by

$$A_a^{(2)} J_a(ka) = - \sum^{n=\infty} \cdot A_n^{(1)}(-1)^a J_{a-a}(k\delta) D_a(ka)$$

that is, $A_a^{(2)} = \dfrac{D_a(ka)}{J_a(ka)} (-1)^a \sum_{n=-\infty}^{n=\infty} A_n^{(1)} J_{a-a}(k\delta)$.

This will again upset the condition on the inner boundary. To satisfy this condition we introduce a fourth function w_3 in a similar manner and so on.

To the order of approximation $w_0+w_1+w_a+w_3$, the periods are given by k which are the roots of the equation

$$\int (w_0+w_1+w_a+w_3) \cos n\theta \, d\theta = c$$

and so on.

The complete solution of problem is then given by

$$w = w_0+w_1+w_a+w_3 + \quad \dots \quad \dots$$

On Electromagnetic Waves due to Electrical Oscillations on the Surface of a Thin Spherical Shell in the Presence of a Non-Concentric Conducting Sphere

BY

Sudhansukumar Banerji

§ 1

In the present paper I propose to consider the effect of *the presence of a perfectly conducting sphere* in the medium on the electrical oscillations started on the surface of a thin spherical shell which is *not concentric* with the sphere. Incidentally, I shall also consider (1) the modifications to be introduced in the method if the sphere be a dielectric one and (2) the method of approximating to the solution in some special cases for the purpose of numerical calculation.

Sir J. J. Thomson[1] has discussed the propagation of electrical oscillations on a thin spherical shell into infinite space and also into the space between the shell and a conducting *concentric* sphere. He has also solved the problem[2] of the scattering of a system of plane electric waves by a metallic sphere.

I wish to express my indebtedness to the writings of Sir J. J. Thomson, Lord Rayleigh[3] and Professors Lamb,[4] Love,[5] Niven,[6] Macdonald[7] and Nicholson[8] on the subject.

[1] J. J. Thomson, "On Electrical Oscillations and the effects produced by the Motion of an Electrified Sphere," *Proc. Lond. Math. Soc.*, Vol. XV, p. 210, (1884), [*Recent Researches in Electricity and Magnetism*, pp. 366-384].

[2] *Recent Researches in Electricity and Magnetism*, pp. 437-451.

[3] Lord Rayleigh, "On the Electromagnetic Theory of Light," *Phil. Mag.*, Vol. XII, p. 81, (1881). [*Scientific Papers*, Part I, p. 518].

[4] Lamb, "On Electrical Motions in a spherical conductor," *Phil. Trans.*, Part II, (1889), p. 519.

[5] Love, "Scattering of Electric Waves by a Dielectric Sphere," *Proc. Lond. Math. Soc*, Vol. XXX, p. 308, (1899).

[6] C. Niven, "On the Induction of Electric Currents in Infinite Plates and Spherical Shells," *Phil. Trans.*, Part II, (1881), p. 307.

[7] Macdonald, *Electric Waves*, Chap. VI.

[8] Nicholson, "The Scattering of Light by a Large Conducting Sphere," *Proc. Lond. Math. Soc.*, Vol. IX, Part I, (1910).

The results obtained by me are believed to be all new.

§ 2

Let A and B be the centres of the shell and the conducting sphere respectively and **a** and **b** their radii.

Measure two parallel systems of Cartesian axes Ax, Ay, Az and Bx', By', Bz' from the two centres, the $x-$ and $x'-$ axes coinciding with the line joining the centres.

Let the polar co-ordinates of any external point be (r, θ, ϕ) referred to A and (r', θ', ϕ) referred to B.

I shall now give some theorems on Bessel Functions which I shall have to use in this paper. The first three of these theorems were given by me in a previous communication.*

If D is the distance between the centres of the two spheres, then

$$r^2 = r'^2 + D^2 + 2r' \, D\cos\theta'.$$

If we write $\cos\theta = \mu$, $\cos\theta' = \mu'$ we have the following theorems :—

(I)†
$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n(\mu) = \sum_{p=0}^{p=\infty} A_{n,p} \frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\mu'), \text{ if } r' < D,$$

where
$$A_{n,p} = \frac{2p+1}{2}(-1)^{n\,p} i \; \underset{a=0}{\text{Lim.}} \int_{-1}^{1} P_n(z) \, P_p(z) \, \phi(a, D, z) \, dz$$

and
$$\phi(a, D, z) = \sqrt{\frac{2\pi}{kD}} \sum_{s=0}^{s=\infty} (-1)^s e^{-s^2 a}(s+\tfrac{1}{2}) K_{s+\frac{1}{2}}(ikD) P_s(z).$$

(II)
$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n(\mu) = \sum_{p=0}^{p=\infty} B_{n,p} \frac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}} P_p(\mu'), \text{ if } r' > D,$$

where
$$B_{n,p} = (2p+1)(-1)^{p+n} \int_{-1}^{1} P_n(z) \, P_p(z) \, e^{ik \, Dz} \, dz.$$

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* *Bulletin of the Calcutta Mathematical Society*, Vol. IV.

† Here the Bessel function of the second kind is defined by the usual relation

$$K_m(ikr) = \frac{\pi e^{-\frac{m\pi i}{2}}}{2 \sin m\pi} \left\} J_{-m}(kr) - e \quad J_m(kr) \right.$$

(III) $\dfrac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}} P_n(\mu) = \overset{p=\infty}{\underset{p=0}{\Sigma}} C_{n,p} \dfrac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\mu'),$

where $C_{n,p} = (2p+1) i^{p-n} \displaystyle\int_{-1}^{1} P_n(z) P_p(z) e^{ik\,Dz}\, dz.$

$\cdot$(IV) $\dfrac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{z}} P_n^m(\mu) {\cos \atop \sin} m\phi = \overset{p=\infty}{\underset{p=0}{\Sigma}} A_{n,p,m} \dfrac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}$

$$P_p^m(\mu') {\cos \atop \sin} m\phi, \text{ if } r' < D,$$

where $A_{n,p,m} = \dfrac{2p+1}{2} \dfrac{(p-m)!}{(p+m)!} (-1)^n i^p \underset{a=0}{\mathrm{Lim.}} \displaystyle\int_{-1}^{1} P_p^m(z) P_n^m(z)$

$$\phi(a, D, z)\, dz.$$

$\dfrac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n^m(\mu) {\cos \atop \sin} m\phi = \overset{p=\infty}{\underset{=0}{\Sigma}} B_{n,p,m} \dfrac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}}$

$$P_p^m(\mu') {\cos \atop \sin} m\phi, \text{ if } r' > D,$$

where $B_{n,p,m} = (2p+1) \dfrac{(p-m)!}{(p+m)!} (-1)^{p+n} \displaystyle\int_{-1}^{1} P_p^m(z) P_n^m(z) e^{ikDz}\, dz.$

(VI) $\dfrac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}} P_n^m(\mu) {\cos \atop \sin} m\phi = \overset{p=\infty}{\underset{p=0}{\Sigma}} C_{n,p,m} \dfrac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}$

$$P_p^m(\mu') {\cos \atop \sin} m\phi,$$

where $C_{n,p,m} = (2p+1) \dfrac{(p-m)!}{(p+m)!} i^{p-n} \displaystyle\int_{-1}^{1} P_p^m(z) P_n^m(z) e^{ikDz}\, dz.$

Proofs of these theorems will be found in the Appendix to this paper.

§ 3.

I shall first consider the case when the shell A encloses the conducting sphere B and an irregular distribution of electricity is produced on the inner surface of the shell A by some cause inside.

Suppose, for simplicity, that a distribution of electricity whose surface density is proportional to the zonal harmonic $P_p(\mu)$ of the pth

degree is produced on the inner surface of the shell A and that the cause producing this distribution is suddenly ⸱removed. Then electrical oscillations will be started on the inner surface of the shell and will be propagated through the medium (which we suppose to be air) enclosed by the shell till they are incident on the surface of the conducting sphere B, when they will be reflected.

The components of magnetic induction (a_0, b_0, c_0) in this system of waves are given by

$$a_0 = A_p^{(o)} \frac{J_{p+\frac{1}{2}}(kr)}{\sqrt{r}} \frac{1}{p} P(\mu) \sin\phi \, e^{ikVt},$$

$$b_0 = -A_p^{(o)} \frac{J_{p+\frac{1}{2}}(kr)}{\sqrt{r}} \frac{1}{p} P(\mu) \cos\phi \, e^{ikVt},$$

$$= 0,$$

where $P_p{}^1(\mu)$ is an associated Legendre function of degree p and order 1, $A_p^{(o)}$ is a constant, V is the velocity of propagation of electromagnetic effects through the medium, and k is determined by the condition, that the tangential electromotive intensity vanishes on the surface of the perfectly conducting shell, and is therefore a root of the equation

$$\frac{d}{da}\left\{ a^{\frac{1}{2}} J_{p+\frac{1}{2}}(ka) \right\} = 0. \tag{1}$$

The components of dielectric polarization (f_0, g_0, h_0) corresponding to these components of magnetic induction are given by

$$f_0 = \frac{-A_p^{(o)}}{4\pi i(2p+1)}\left[p \frac{J_{p+\frac{1}{2}}(kr)}{\sqrt{r}} P^1_{p+1}(\mu) \cos\phi - (p+1) \frac{J_{p-\frac{1}{2}}(kr)}{\sqrt{r}} P^1_{p-1}(\mu) \cos\phi \right] e^{ikVt},$$

$$g_0 = -\frac{A_p^{(o)}}{4\pi i(2p+1)}\left[p \frac{J_{p+\frac{1}{2}}(kr)}{\sqrt{r}} P^1_{p+1}(\mu) \sin\phi - (p+1) \frac{J_{p-\frac{1}{2}}(kr)}{\sqrt{r}} P^1_{p-1}(\mu) \sin\phi \right] e^{ikVt},$$

$$h = -\frac{A_p^{(o)}(p+1)}{4\pi i(2p+1)} p \left[\frac{J_{p+\frac{1}{2}}(kr)}{\sqrt{r}} P_{p+1}(\mu) + \frac{J_{p-\frac{1}{2}}(kr)}{\sqrt{r}} P_{p-1}(\mu) \right] e^{ikVt},$$

If the surface distribution, instead of being expressed by a single zonal harmonic, is expressed by a series of zonal harmonics no essential difference would be introduced in the method; the effect would be simply summational.

§ 4.

Let (a_1, b_1, c_1) and (f_1, g_1, h_1) be the components of magnetic induction and dielectric polarization respectively in the disturbance reflected from the conducting surface B. In determining these components we proceed as if the shell A were absent from the medium.

Prof. Lamb* has shown that the most general related solutions of the same system of differential equations

$$(\nabla^2 + k^2)\, a = 0, \quad (\nabla^2 + k^2)\, b = 0, \quad (\nabla^2 + k^2)\, c = 0,$$

which also satisfy the solenoidal condition

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} = 0,$$

fall into two types.

For waves which are being propagated outwards, the solutions of the first type are given by

$$(a, b, c) = \frac{K_{n+\frac{1}{2}}(ikr)}{r^{n+\frac{1}{2}}} \left(y\, \frac{\partial}{\partial z} - z\, \frac{\partial}{\partial y}, \; z\, \frac{\partial}{\partial x} - x\, \frac{\partial}{\partial z}, \; x\, \frac{\partial}{\partial y} - y\, \frac{\partial}{\partial x} \right) w_n,$$

where w_n is a spherical solid harmonic of positive degree n, and the solutions of the second type are given by

$$= -\frac{k}{i} \left[(n+1) \frac{K_{n-\frac{1}{2}}(ikr)}{r^{n-\frac{1}{2}}}\, \frac{\partial w'}{\partial x} \;,\; n \frac{K_{n+\frac{3}{2}}(ikr)}{r^{-n-\frac{3}{2}}}\, \frac{\partial}{\partial x}\, \frac{w'_n}{r^{2n+1}} \right],$$

where w'_n is another spherical solid harmonic of positive degree n. The corresponding expressions for b and c are obtained from this by changing x to y and z respectively.

Combining these two types we see that the most general expressions for (a, b, c) can be written in the form

$$a = -\sum \frac{k}{i} \left[(n+1) \frac{K_{n-\frac{1}{2}}(ikr)}{r^{n-\frac{1}{2}}}\, \frac{\partial w'_n}{\partial x} + n \frac{K_{n+\frac{1}{2}}(ikr)}{r^{-n-\frac{3}{2}}}\, \frac{\partial}{\partial x}\, \frac{w'_n}{r^{2n+1}} \right] e^{ikVt}$$

$$\frac{K_{n+\frac{1}{2}}(ikr)}{r^{n+\frac{1}{2}}} \left(y\, \frac{\partial}{\partial z} - z\, \frac{\partial}{\partial y} \right) w_n e^{ikVt}$$

where the summation refers to the different orders of the harmonics. The corresponding expressions for b and c are obtained from this by cyclical interchange of the letters x, y, z.

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* *Proc. Lond. Math. Soc.* Vol. XIII, p. 189, (1881).

Again, from the relation

$$\frac{4\pi}{V} \frac{\partial}{\partial t} (f, g, h) = \text{curl } (a, \beta, \gamma),$$

where (a, β, γ) are the components of magnetic force which are identical in air with the components of magnetic induction, we see that the most general expressions for (f, g, h) can be written in the form

$$4\pi i k f = -\Sigma \frac{k}{(2n+1)i} \left[(n+1) \frac{K_{n-\frac{1}{2}}(ikr)}{r^{n-\frac{1}{2}}} \frac{\partial w_n}{\partial x} + n \frac{K_{n+\frac{1}{2}}(ikr)}{r^{-n-\frac{3}{2}}} \right.$$
$$\left. \frac{\partial}{\partial x} \frac{w_n}{r^{2n+1}} \right] e^{ikVt}$$
$$+ \Sigma k^2 (2n+1) \frac{K_{n+\frac{1}{2}}(ikr)}{r^{n+\frac{1}{2}}} \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) w'_n e^{ikVt},$$

where w_n and w'_n are the same two solid harmonics. The expressions for g and h are obtained from this by cyclical interchange of the letters x, y, z.

For waves, which converge to the origin, the most general expressions for (a, b, c) and (f, g, h) can be written in the forms

$$a = -\Sigma k \left[(n+1) \frac{J_{n-\frac{1}{2}}(kr)}{r^{n-\frac{1}{2}}} \partial w'_n - n \frac{J_{n+\frac{1}{2}}(kr)}{r^{-n-\frac{3}{2}}} \frac{\partial}{\partial x} \frac{w'_n}{r^{2n+1}} \right] e^{ikVt}$$
$$+ \Sigma \frac{J_{n+\frac{1}{2}}(kr)}{r^{n+\frac{1}{2}}} \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) w_n e^{ikVt},$$

etc.

$$4\pi i k f = -\Sigma \frac{k}{2n+1} \left[(n+1) \frac{J_{n-\frac{1}{2}}(kr)}{r^{n-\frac{1}{2}}} \frac{\partial w}{\partial x} - n \frac{J_{n+\frac{1}{2}}(kr)}{r^{-n-\frac{3}{2}}} \frac{\partial}{\partial x} \frac{w_n}{r^{2n+1}} \right] e^{ikVt}$$
$$+ \Sigma k^2 (2n+1) \frac{J_{n+\frac{1}{2}}(kr)}{r^{n+\frac{1}{2}}} \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) w'_n e^{ikVt}$$

etc.

We, therefore, assume the following expressions for (a_1, b_1, c_1) and (f_1, g_1, h_1) in the disturbance reflected from the conducting surface B—

$$a_1 = -\Sigma \frac{k}{i} \left[(n+1) \frac{K_{n-\frac{1}{2}}(ikr')}{r'^{n-\frac{1}{2}}} \frac{\partial w'_n}{\partial x'} + n \frac{K_{n+\frac{1}{2}}(ikr')}{r'^{-n-\frac{3}{2}}} \frac{\partial}{\partial x'} \frac{w'_n}{r'^{2n+1}} \right] e^{ikVt}$$

$$+ \Sigma \frac{n+\frac{1}{3}}{n+\frac{1}{3}} \frac{(ikr')}{} \left(y' \frac{\partial}{\partial z'} - z' \frac{\partial}{\partial y'} \right) w_n e^{ikVt},$$

etc.

$$4\pi ikf_1 = -\Sigma \frac{k}{(2n+1)i} \left[(n+1) \frac{K_{n-\frac{1}{2}}(ikr')}{r'^{n-\frac{1}{2}}} \frac{\partial w_n}{\partial z'} + n \frac{K_{n+\frac{3}{2}}(ikr')}{r'^{-n-\frac{3}{2}}} \right.$$

$$\left. \frac{\partial}{\partial z'} \frac{w_n}{r'^{2n+1}} \right] e^{ikVt}$$

$$+ \Sigma k^2 (2n+1) \frac{K_{n+\frac{1}{2}}(ikr')}{r'^{n+\frac{1}{2}}} \left(y' \frac{\partial}{\partial z'} - z' \frac{\partial}{\partial y'} \right) w'_n e^{ikVt}.$$

etc.,

the origin of co-ordinates being now the centre of the sphere B.

We proceed now to determine the arbitrary solid harmonics w_n and w'_n so as to satisfy the boundary conditions on the surface of the sphere B.

The boundary conditions can be expressed as follows:—

(I) Since the sphere B is a perfect conductor, the electric force, and therefore the electric polarization, is at right angles to its surface. Hence if R is the normal electric polarization, Θ that along a tangent to a meridian, Φ that along a parallel of latitude, then the condition

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0$$

is equivalent to

$$\frac{\partial}{\partial r'} \left(r'^2 R \right) + \frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \left(r' \sin \theta' \Theta \right) + \frac{1}{\sin \theta'} \frac{\partial}{\partial \phi} \left(r' \Phi \right) = 0.$$

but since Θ and Φ vanish all over the sphere, this gives the condition

$$\frac{\partial}{\partial r'} \left(r'^2 R \right) = 0, \text{ when } r' = b,$$

where $r'^2 R = x' (f_1 + f_o) + y' (g_1 + g_o) + z' (h_1 + h_o)$.

(II) We have also the condition that the radial magnetic induction vanishes all over the surface of the sphere. Hence

$$x' (a_1 + a_o) + y' (b_1 + b_o) + z' (c_1 + c_o) = 0, \text{ when } r' = b.$$

Now, we have

$$4\pi ik (x' f_1 + y' g_1 + z' h_1) = \Sigma - \frac{kn (n+1)}{(2n+1)i} \left\{ \frac{K_{n-\frac{1}{2}}(ikr')}{r'^{n-\frac{1}{2}}} \right.$$

$$-r'^2 \left. \frac{K_{n+\frac{3}{2}}(ikr')}{r'^{n+\frac{3}{2}}} \right\} w_n e^{ikVt} = -\Sigma \, n(n+1) \frac{K_{n+\frac{1}{2}}(ikr')}{r'^{n+\frac{1}{2}}}$$

$$w_n e^{ikVt}.$$

Also we have by theorem (VI) given in §2,

$$a_o = A_p \overset{(o)n=\infty}{\underset{n=0}{\Sigma}} C_{p,\,n,\,1} \frac{J_{n+\frac{1}{2}}(kr')}{\sqrt{r'}} P_n^{1}(\mu') \sin \phi \, e^{ikVt}$$

$$b_o = -A_x \overset{(o)\; n=\infty}{\underset{n=0}{\Sigma}} C_{p,\,n,\,1} \frac{J_{n+\frac{1}{2}}(kr')}{\sqrt{r'}} P_n^{1}(\mu') \cos \phi \, e^{ikVt}$$

$$c_o = 0,$$

where $C_{p,\,n,\,1} = \dfrac{2n+1}{n(n+1)} i^{n-p} \displaystyle\int_{-1}^{1} P_p^{1}(z) \, P_n^{1}(z) e^{ikDz} \, dz.$

We can write these expressions in the forms

$$a_o = A_p \overset{(o)\; n=\infty}{\underset{0}{\Sigma}} C_{p,\,n,\,1} \frac{J_{n+\frac{1}{2}}(kr')}{r'^{n+\frac{1}{2}}} \left(y' \frac{\partial}{\partial z'} - z' \frac{\partial}{\partial y'} \right) \phi_n \, e^{ikVt}$$

$$b_n = A_x \overset{(o)\; n=\infty}{\underset{n=0}{\Sigma}} C_{p,\,n,\,1} \frac{J_{n+\frac{1}{2}}(kr')}{r'^{n+\frac{1}{2}}} \left(z' \frac{\partial}{\partial r'} - x' \frac{\partial}{\partial z'} \right) \phi_n \, e^{ikVt}$$

$$C_o = A_p \overset{(o)\; n=\infty}{\underset{n=0}{\Sigma}} C_{p,\,n,\,1} \frac{J_{n+\frac{1}{2}}(kr')}{r'^{n+\frac{1}{2}}} \left(x' \frac{\partial}{\partial y'} - y' \frac{\partial}{\partial z'} \right) \phi_n \, e^{ikVt}$$

where $\phi_n = r'^n P_n(\mu')$.

The corresponding expressions for (f_o, g_o, h_o) can be written in the form

$$4\pi \, ik f_o = -A_p \overset{(o)}{} \frac{k \, C_{p,\,n,\,1}}{2n+1} \left[(n+1) \frac{J_{n-\frac{1}{2}}(kr')}{r'^{n-\frac{1}{2}}} \frac{\partial \phi_n}{\partial x'} \right.$$

$$\left. -n \frac{J_{n+\frac{3}{2}}(kr')}{r'^{-n-\frac{3}{2}}} \frac{\partial}{\partial x'} \frac{\phi_n}{r'^{2n+1}} \right] e^{ikVt}$$

The expressions for g_o and h_o are obtained from this by changing x' to y' and z respectively.

Thus, we get

$$4\pi\, ik\, (x'f_o + y'\, g_o + z'\, h_o) = -A_p^{(o)} \sum n\, (n+1)\, C_{p,\,n},$$

$$\frac{J_{n+\frac{1}{2}}(kr')}{r'^{n+\frac{1}{2}}}\, \phi_n\, e^{ikVt}.$$

On substitution in the boundary condition

$$\frac{\partial}{\partial r'}\, (r'^2\, R) = 0, \text{ when } r' = b.$$

we get

$$w_n = -A_p^{(o)}\, r'^n\, P_n(\mu')\, \frac{\dfrac{d}{db}\left[b^2\, J_{n+\frac{1}{2}}(kb)\right]}{\dfrac{d}{db}\left[b^{\frac{1}{2}}\, K_{n+\frac{1}{2}}(ikb)\right]}\, C_{p,\,n,\,1}.$$

I write this in the form

$$w_n = A_n^{(1)}\, r'^n\, P_n(\mu').$$

Also, we have

$$x'\, a_1 + y'\, b_1 + z'\, c_1 = \sum n\, (n+1)\, (2n+1)\, \frac{K_{n+\frac{1}{2}}(ikr')}{r'^{n+\frac{1}{2}}}$$

$$w_n'\, e^{ik\,Vt},$$

and $x'\, a_o + y'\, b_o + z'\, c_o = 0.$

Therefore, by condition (11), $w_n' = 0.$

§ 6.

Thus we see that the expressions for (a_1, b_1, c_1) and (f_1, g_1, h_1) reduce to the simple forms

$$a_1 = \sum_{n=0}^{n=\infty} A_n^{(1)}\, \frac{K_{n+\frac{1}{2}}(ikr')}{\sqrt{r'}}\, P_n^1(\mu')\, \sin\phi\, e^{ikVt},$$

$$b_1 = -\sum_{n=0}^{n=\infty} A_n^{(1)}\, \frac{K_{n+\frac{1}{2}}(ikr')}{\sqrt{r'}}\, P_n^{\cdot}(\mu')\, \cos\phi\, e^{ikVt}$$

$$c_1 = 0.$$

$$f_1 = \frac{1}{4\pi} \sum_{n=0}^{n=\infty} \frac{A_n^{(1)}}{2n+1} \left[n \frac{K_{n+\frac{3}{2}}(ikr')}{\sqrt{r'}} P_{n+1}^1(\mu') \cos\phi \right.$$

$$\left. + (n+1) \frac{K_{n-\frac{1}{2}}(ikr')}{\sqrt{r'}} P_{n-1}^1(\mu') \cos\phi \right] ikVt$$

$$g_1 = \frac{1}{4\pi} \sum_{n=0}^{n=\infty} \frac{A_n^{(1)}}{2n+1} \left[n \frac{K_{n+\frac{3}{2}}(ikr')}{\sqrt{r'}} P_{n+1}^1(\mu') \sin\phi \right.$$

$$\left. + (n+1) \frac{K_{n-\frac{1}{2}}(ihr')}{\sqrt{r'}} P_{n-1}^1(\mu') \sin\phi \right] e^i$$

$$h_1 = \frac{1}{4\pi} \sum_{u=0}^{n=\infty} \frac{n(n+1)A_n^{(1)}}{2n+1} \left[\frac{K_{n+\frac{3}{2}}(ikr')}{\sqrt{r'}} P_{n+1}(\mu') \right.$$

$$\left. \frac{K_{n-\frac{1}{2}}(ihr')}{\sqrt{r'}} P_{n-1}(\mu') \right] ikVt$$

where

$$A_n^{(1)} = -A_p^{(0)} \frac{\frac{d}{db}\left[b^{\frac{1}{2}} J_{n+\frac{1}{2}}(hb) \right]}{\frac{d}{db}\left[b^{\frac{1}{2}} K_{n+1}(ikb) \right]} \frac{2n+1}{n(n+1)} i^{-p+n} \int_1^1 P_p^1(z) P_n^1(z)$$

$$ikD:$$

We see that the expressions (a_1, b_1, c_1) belong to the first type of solutions and (f_1, g_1, h_1) to the second type, that is, they are of the same type as the initial disturbance started on the surface of the shell A.

The periods of oscillation are no longer determined by k which are the roots of the equation (1). To the order of approximation represented by $(u_0+a_1, b_0+b_1, c_0+c_1)$ and $(f_0+f_1, g_0+g_1, h_0+h_1)$ the periods are easily seen to be given by k which are the roots of the equation

$$\int_{-1}^1 \frac{\partial}{\partial r} (r^2 R) \cdot P_n(\mu) d\mu \quad \begin{array}{c} =0, \\ r=a \end{array}$$

where $rR = v(f_1+f_0)+y(g_1+g_0)+z(h_1+h_0)$, that is to say, by k which are the roots of the equation

$$\frac{d}{da}\left[a^{\frac{1}{2}}J_{n+\frac{1}{2}}(la)\right] - \frac{d}{da}\left[a^{\frac{1}{2}}K_{n+\frac{1}{2}}(ila)\right]\frac{\dfrac{d}{db}\left[b^{\frac{1}{2}}J_{n+\frac{1}{2}}(lb)\right]}{\dfrac{d}{db}\left[b^{\frac{1}{2}}K_{n+\frac{1}{2}}(ilb)\right]}$$

$$\sum_{p=0}^{p=\infty} C_{p,n,1}\, B_{p,n,1} = 0.$$

§ 7.

Waves of magnetic induction and dielectric polarization determined by (a_1, b_1, c_1) and (f_1, g_1, h_1) will be incident on the surface of the shell A and will undergo a second reflection. We denote the components of magnetic induction and dielectric polarization in this second reflected disturbance by (a_2, b_2, c_2) and (f_2, g_2, h_2) respectively.

We now assume the following expressions for (a_2, b_2, c_2) and (f_2, g_2, h_2),

$$a_2 = -\sum k\left[(n+1)\frac{J_{n-\frac{1}{2}}(lr)}{r^{n-\frac{1}{2}}}\cdot\frac{\partial w'_n}{\partial x} - n\frac{J_{n+\frac{3}{2}}(lr)}{r^{-n-\frac{3}{2}}}\cdot\frac{\partial}{\partial x}\frac{w'_n}{2n+1}\right]e^{ikVt}$$

$$+\sum\frac{J_{n+\frac{1}{2}}(lr)}{r^{n+\frac{1}{2}}}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)w_n e^{ikVt}$$

etc.

$$4\pi il f_2 = -\sum\frac{k}{2n+1}\left[(n+1)\frac{J_{n-\frac{1}{2}}(kr)}{r^{n-\frac{1}{2}}}\frac{\partial w_n}{\partial x} - n\frac{J_{n+\frac{3}{2}}(lr)}{r^{-n-\frac{3}{2}}}\frac{\partial}{\partial x}\frac{w_n}{r^{2n+1}}\right]e^{ikVt}$$

$$+\sum k^2(2n+1)\frac{J_{n+\frac{1}{2}}(kr)}{r^{n+\frac{1}{2}}}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)w_n' e^{ikVt},$$

Since the expressions for (a_1, b_1, c_1), etc., are of the same type as the expressions for (a_0, b_0, c_0), etc., we proceed exactly in the same manner as in §§ 5 and 6 to satisfy the boundary conditions on the surface of the sphere A. Here, however, we use the theorem (V), instead of the theorem (VI) used in those articles.

We find as before

$$w_n' = 0$$

and
$$w_n = A_n^{(2)} r^n P_n(\mu),$$

where
$$A_n^{(2)} = -\frac{\dfrac{d}{da}\left[a^{\frac{1}{2}} K_{n+\frac{1}{2}}(i l a)\right]}{\dfrac{d}{da}\left[a^{\frac{1}{2}} J_{n+\frac{1}{2}}(k a)\right]} \sum_{p=0}^{p=\infty} B_{p,\,n,\,1} A_p^{(1)}$$

i.e.,
$$A_n^{(2)} = -\frac{\dfrac{d}{da}\left[a^{\frac{1}{2}} K_{n+\frac{1}{2}}(i l a)\right]}{\dfrac{d}{da}\left[a^{\frac{1}{2}} J_{n+\frac{1}{2}}(k a)\right]} \frac{2n+1}{n(n+1)} (-1)^n \int_{-1}^{1} e^{ikDz}$$

$$P_n^1(z) \left[\sum_{p=0}^{p=\infty} A_p^{(1)} (-1)^p P_p^1(z)\right] dz.$$

Thus, the expressions for (a_2, b_2, c_2) and (f_2, g_2, h_2) can be written in the forms

$$a_2 = \sum_{n=0}^{n=\infty} A_n^{(2)} \frac{J_{n+\frac{1}{2}}(lr)}{\sqrt{r}} P_n^1(\mu) \sin\phi \, e^{ikVt},$$

$$b_2 = -\sum_{n=0}^{n=\infty} A_n^{(2)} \frac{J_{n+\frac{1}{2}}(lr)}{\sqrt{r}} P_n^1(\mu) \cos\phi \, e^{ikVt},$$

$$c_2 = 0,$$

$$f_2 = -\frac{1}{4\pi i} \sum_{n=c}^{n=\infty} \frac{A_n^{(2)}}{2n+1} \left[n \frac{J_{n+\frac{3}{2}}(lr)}{\sqrt{r}} P_{n+1}^1(\mu) \cos\phi \right.$$
$$\left. -(n+1) \frac{J_{n-\frac{1}{2}}(lr)}{\sqrt{r}} P_{n-1}^1(\mu) \cos\phi \right] e^{ikVt},$$

$$g_2 = -\frac{1}{4\pi i} \sum_{n=0}^{n=\infty} \frac{A_n^{(2)}}{2n+1} \left[n \frac{J_{n+\frac{3}{2}}(lr)}{\sqrt{r}} P_{n+1}^1(\mu) \sin\phi \right.$$
$$\left. -(n+1) \frac{J_{n-\frac{1}{2}}(kr)}{\sqrt{r}} P_{n-1}(\mu) \sin\phi \right] e^{ikVt},$$

$$h_2 = -\frac{1}{4\pi i} \sum_{n=0}^{n=\infty} \frac{n(n+1) A_n^{(2)}}{2n+1} \left[\frac{J_{n+\frac{3}{2}}(lr)}{\sqrt{r}} P_{n+1}(\mu) + \frac{J_{n-\frac{1}{2}}(lr)}{\sqrt{r}} \right.$$
$$\left. P_{n-1}(\mu) \right] e^{ikVt}.$$

§ 8.

In this way an infinite number of reflections will occur alternately on the two surfaces and it is easy to determine by the method indicated above the components of magnetic induction and dielectric polarization in all these reflected disturbances.

If now (a, b, c) and (f, g, h) denote the components of magnetic induction and dielectric polarization in the total disturbance in the medium between the two spherical surfaces, we have

$$a = a_0 + a_1 + a_2 + a_3 + \text{etc.},$$

etc.,

$$f = f_0 + f_1 + f_2 + f_3 + \text{etc.},$$

etc.,

The periods of the oscillation are given by k which are the roots of the equation

$$\left| \int_{-1}^{1} \frac{\partial}{\partial r} (r^2 R) \cdot P_n(\mu) \, d\mu \right|_{r=a} = 0,$$

where $rR = x (f_0 + f_1 + \text{etc.}) + y (g_0 + g_1 + \text{etc.}) + z (h_0 + h_1 + \text{etc.})$.

For, from the manner in which the components a_0, a_1, a_2, etc, f_0, f_1, f_2, etc., have been obtained it is obvious that the disturbance determined by the magnetic induction (a, b, c) and dielectric polarization (f, g, h) satisfies the boundary conditions over the surfaces of both the spheres and is due to the oscillations of a distribution of electricity on the inner surface of the shell A whose initial surface density varies as the zonal harmonic $P_p(\mu)$.

Moreover, from the fact that the expressions for (a_0, b_0, c_0) (a_1, b_1, c_1), (a_2, b_2, c_2), etc., are all of the same type we at once deduce the following property :—

The induced current on the surface of the sphere B is of the same type as the inducing current on the surface of the sphere A. For instance,

(1) The inducing current on the surface of the sphere A flows meridionally having the line joining the centres as axis and so does the induced current on the surface of the sphere B.

(2) The lines of magnetic force on both the spheres are a series of small circles having the line joining the centres as axis.

If the distribution of electricity on the surface of the sphere A instead of being symmetrical about the line joining the centres is symmetrical about any other line we could effect the solution with the help of the theorems given in §2, by transforming the zonal harmonic to harmonics referred to the line joining the centres.

§ 9.

Suppose now that the radius of the conducting sphere B is very small in comparison with the radius of the shell A. In this case the series expressing the solution given in § 8, is very rapidly converging and we can easily approximate to the solution.

For when b is small, we have approximately

$$(kb)^{-\frac{1}{2}} \; J_{n+\frac{1}{2}}(kb) = \sqrt{\frac{2}{\pi}} \cdot \frac{(kb)^n}{1 \cdot 3 \dots (2n+1)},$$

$$(lb)^{-\frac{1}{2}} \; K_{n+\frac{1}{2}}(ilb) = \sqrt{i} \cdot i^{n+1}(-1)^n \; \sqrt{\frac{\pi}{2}} \cdot 1 \cdot 3 \dots (2n-1) \cdot$$

$$\frac{-ilb}{(lb)^{n+1}}$$

Therefore

$$\frac{\frac{d}{db}\left[b^{\frac{1}{2}} J_{n+\frac{1}{2}}(lb) \right]}{\frac{d}{db}\left[b^{\frac{1}{2}} K_{n+\frac{1}{2}}(ilb) \right]} = \frac{2 i^{n+1}}{\pi \sqrt{i}} \frac{n+1}{n(2n+1)} \frac{(lb)^{2n+1}}{\{1 \cdot 3 \dots (2n-1)\}^2} \quad ilb$$

Thus, in this case, we get

$$A_n^{(1)} = -A_p^{(0)} \frac{2(-1)^n i^{-p}}{\pi \sqrt{i} \, n^2} \frac{(kb)^{2n+1}}{\{1 \cdot 3 \dots (2n-1)\}^2} e^{ilb} \int_{-1}^{1} (1-z^2)$$

$$\frac{dP_p(z)}{dz} \cdot \frac{dP_n(z)}{dz} e^{ilDz} \, dz.$$

Also, if a is large, we have approximately

$$K_{n+\frac{1}{2}}(ika) = \sqrt{\frac{\pi}{2ka}} e^{-i\left(ka + \frac{\pi}{4}\right)}$$

and

$$J_{n+\frac{1}{2}}(ka) = \sqrt{\frac{2}{\pi ka}} \cos\left\{\frac{(n+1)\pi}{2} - ka\right\}.$$

Therefore,

$$\frac{\dfrac{d}{da}\left[a^{\frac{1}{2}} K_{n+\frac{1}{2}}(i/a) \right]}{\dfrac{d}{da}\left[a^{\frac{1}{2}} J_{n+\frac{1}{2}}(la) \right]} = \frac{\pi i}{2} \cdot \frac{e^{-i\left(la + \frac{\pi}{4} \right)}}{\cos\left\{ \dfrac{(n+1)}{2}\dfrac{\pi}{2} - la \right\}}.$$

Thus we get

$$A_n^{(2)} = -\frac{\pi i}{2}\frac{2n+1)}{n(n+1)}\ (-1)^n \frac{e^{-i\left(la + \frac{\pi}{4} \right)}}{\cos\left\{ \dfrac{(n+1)\pi}{2} - la \right\}} \int_{-1}^{1}(1-z^2)\, e^{ikDz}$$

$$\frac{dP_n(z)}{dz}\left\{ \sum_{p=0}^{p=\infty} (-1)^p A_p^{(1)} \frac{dP_p(z)}{dz} \right\} dz.$$

So that if we take account of one or two reflected disturbances we shall usually get a sufficient approximation to the solution.

If, on the other hand, the distance between the centres of the two spheres is also small, so that we can neglect all powers of D above the first, we have

$$e^{ikDz} = 1 + ikDz.$$

In this case the integral $\displaystyle\int_{-1}^{1} e^{ikDz}\, P_n^1(z)\, P_p^1(z)\, dz$, has the values

$$ikD.\ \frac{2p'p-1)'p+1)}{(2p-1,(2p+1)},\quad \frac{2p(p+1)}{2p+1}\ \text{and}\ ikD.\ \frac{2p'p+1)(p+2)}{(2p+1)\,2p+3)}\ \text{when}\ n=$$

$p-1$, p and $p+1$ respectively and for other values of n it vanishes.

So that in this case all the co-efficients other than $A_{p-1}^{(1)}$, $A_p^{(1)}$ and $A_{p+1}^{(1)}$ vanish.

Similarly we can treat the co-efficients $A_n^{(2)}$'s.

§ 10.

If the sphere B be a dielectric one, we shall have to modify our solution. In this case a part of the waves of magnetic induction and dielectric polarization determined by (a_0, b_0, c_0) and (f_0, g_0, h_0) on incidence on the surface of the sphere B will enter into the body of the sphere and a part will be reflected.

We denote the components of magnetic induction and polarization in the reflected disturbance by (a_1, b_1, c_1) and (f_1, g_1, h_1) and in the disturbance which enters the body of the sphere by (a_1', b_1', c_1') and (f_1', g_1', h_1') respectively.

Since (a_1, b_1, c_1) and (f_1, g_1, h_1) determine a wave system which is being propagated outwards and (a_1', b_1', c_1'), (f_1', g_1', h_1') one which is converging towards the centre of the sphere B, we can assume the following expressions for these components :—

$$a_1 = -\sum \frac{k}{i}\left[(n+1)\frac{K_{n-\frac{1}{2}}(ikr')}{r'^{n-\frac{1}{2}}}\frac{\partial \chi_n'}{\partial_{\omega'}} + n\frac{K_{n+\frac{3}{2}}(ilr')}{r'^{-n-\frac{1}{2}}}\frac{\partial}{\partial_{\omega'}}\frac{\chi_n'}{r'^{2n+1}}\right]^{ikVt}$$

$$+\sum \frac{K_{n+\frac{1}{2}}(ikr')}{r'^{n+\frac{1}{2}}}\left(y'\frac{\partial}{\partial_{\omega'}} - z'\frac{\partial}{\partial y'}\right)\phi_n'^{ikVt}$$

etc.,

$$4\pi i h f_1 = -\sum \frac{k}{(2n+1)i}\left[(n+1)\frac{K_{n-\frac{1}{2}}(ilr')}{r'^{n-\frac{1}{2}}}\frac{\partial \phi_n'}{\partial_{\omega'}} + n\frac{K_{n+\frac{3}{2}}(ihr')}{r'^{-n-\frac{3}{2}}}\right.$$

$$\left.\frac{\partial}{\partial_{\omega'}}\frac{\phi_n'}{r'^{2n+1}}\right]e^{ikVt} + \sum k^2(2n+1)\frac{K_{n+\frac{1}{2}}(ilr')}{r'^{n+\frac{1}{2}}}\left(y'\frac{\partial}{\partial_{\omega'}} - z'\frac{\partial}{\partial y'}\right)\chi_n'^{ikVt}$$

etc.

where ϕ_n' and χ_n' are two arbitrary solid harmonics of degree n.

And

$$a_1' = -\sum k'\left[(n+1)\frac{J_{n-\frac{1}{2}}(l'r')}{r'^{n-\frac{1}{2}}}\frac{\partial \chi_n''}{\partial_{\omega'}} - n\frac{J_{n+\frac{3}{2}}(l'r')}{r'^{-n-\frac{3}{2}}}\frac{\partial}{\partial_{\omega'}}\frac{\chi_n''}{r'^{2n+1}}\right]^{ikVt}$$

$$+\sum \frac{J_{n+\frac{1}{2}}(l'r')}{r'^{n+\frac{1}{2}}}\left(y'\frac{\partial}{\partial_{\omega'}} - z'\frac{\partial}{\partial y'}\right)\phi_n''e^{ikVt}$$

etc.,

$$4\pi i k r f_1' = -\sum \frac{k'}{(2n+1)}\left[(n+1)\frac{J_{n-\frac{1}{2}}(k'r')}{r'^{n-\frac{1}{2}}}\frac{\partial \phi_n''}{\partial_{\omega'}} - n\frac{J_{n+\frac{3}{2}}(h'r')}{r'^{-n-\frac{3}{2}}}\right.$$

$$\left.\frac{\phi_n''}{r'^{2n+1}}\right]e^{ikVt} + \sum k'^2(2n+1)\frac{J_{n+\frac{1}{2}}(l'r')}{r'^{n+\frac{1}{2}}}\left(y'\frac{\partial}{\partial_{\omega'}} - z'\frac{\partial}{\partial y'}\right)\chi_n''e^{ikVt}$$

where ϕ_n'', χ_n'' are two arbitrary solid harmonics, and l' is different from k and is given by

$$l'^2 = h^2 K r,$$

where K is the specific inductive capacity and τ the magnetic permeability of the dielectric sphere B.

Moreover, the components (a_0, b_0, c_0) and (f_0, g_0, h_0) of the incident wave system when transferred to the centre of the sphere B can be written in the forms

$$a_0 = A_p \overset{(^0)\, n=\infty}{\underset{n=0}{\sum}} C_{p,\,n,\,1} \frac{J_{n+\frac{1}{2}}(l\,r')}{r'^{n+\frac{1}{2}}} \left(y' \frac{\partial}{\partial \jmath} - z' \frac{\partial}{\partial y'} \right) \phi_n \quad e^{i\hbar Vt}$$

etc.,

$$4\pi i l f_0 = -A_p \, k \overset{(^0)\, n=\infty}{\underset{n=0}{\sum}} \frac{C_{p,n,1}}{2n+1} \left[(n+1) \frac{J_{n-\frac{1}{2}}(l r')}{r'^{n-\frac{1}{2}}} \frac{\partial \phi_n}{\partial \jmath} - n \frac{J_{n+\frac{3}{2}}(l r')}{r'^{-n-\frac{3}{2}}} \right.$$

$$\left. \frac{\phi_n}{z'} \frac{1}{r'^{2n+1}} \right] e^{i\hbar Vt} \quad \text{where } \phi_n = r' P_n(\mu').$$

Now the tangential components of the magnetic and electric forces on the surface of the dielectric sphere must be continuous.

Confining our attention to the components (a_1, b_1, c_1), we see that the expressions which we have assumed for these components consist of two types of solutions; the first type is

$$\frac{K_{n+\frac{1}{2}}(ikr')}{r'^{n+\frac{1}{2}}} \left(y' \frac{\partial}{\partial \jmath} - z' \frac{\partial}{\partial y'} \right) \phi'_n, \text{ etc.;}$$

the second type is

$$-\frac{k}{i} \left[(n+1) \frac{K_{n-\frac{1}{2}}(ikr')}{r'^{n-\frac{1}{2}}} \frac{\partial v'_n}{\partial \jmath} + n \frac{K_{n+\frac{3}{2}}(ikr')}{r'^{-n-\frac{1}{2}}} \frac{\partial}{\partial \jmath} \frac{x'_n}{r'^{2n+1}} \right]$$

The vector given by the solutions of the first type is wholly transverse. To determine the transverse components of the vector given by the solutions of the second type we subtract from each of the components of this vector the corresponding resolved part of the radial component of the vector, i.e., the quantit es

$$-n(n+1)(2n+1)\frac{z'}{r'^2} \frac{K_{n+\frac{1}{2}}(ikr')}{r'^{n+\frac{1}{2}}} \chi'_n, \text{ etc,}$$

so that the contributions to the transverse component by the components of the vector are (after a little reduction) given by expressions of the type

$$\left\{ n\, \frac{K_{n+\frac{1}{2}}(ikr')}{r'^{n+\frac{1}{2}}} + ki\, \frac{K_{n-\frac{1}{2}}(ikr')}{r'^{n-\frac{1}{2}}} \right\} \left\{ (n+1)\, \frac{\partial \chi'_n}{\partial x'} \right.$$

$$\left. + nr'^{2n+1} \frac{\partial}{\partial x'} \frac{\chi'_n}{r'^{2n+1}} \right\}.$$

Similarly, the contributions to the transverse component by the components of the solution of the second type in (a'_1, b'_1, c'_1) are expressions of the type

$$\left\{ n\, \frac{J_{n+\frac{1}{2}}(k'r')}{r'^{n+\frac{1}{2}}} - k'\, \frac{J_{n-\frac{1}{2}}(k'r')}{r'^{n-\frac{1}{2}}} \right\} \left\{ (n+1)\, \frac{\partial \chi''_n}{\partial x'} \right.$$

$$\left. + nr'^{2n+1} \frac{\partial}{\partial x'} \frac{\chi''_n}{r'^{2n+1}} \right\}.$$

The continuity of tangential magnetic force on the surface of the sphere B, now gives the following conditions :—

$$\overset{(\circ)}{A_p}\, C_{p,n,1}\, J_{n+\frac{1}{2}}(lb)\phi_n + K_{n+\frac{1}{2}}'(ilb)\, \phi'_n = \frac{1}{\tau}\, J_{n+\frac{1}{2}}(k'b)\phi''_n \qquad (1)$$

$$\left\{ n\, K_{n+\frac{1}{2}}(ilb) + lbi\, K_{n-\frac{1}{2}}(ilb) \right\} \chi'_n$$

$$= \frac{1}{\tau} \left\{ nJ_{n+\frac{1}{2}}(k'b) - k'bJ_{n-\frac{1}{2}}(k'b) \right\} \chi''_n \qquad (2)$$

Similarly, the continuity of tangential electric force on the surface of the sphere B, gives the following conditions :—

$$k^2\, K_{n+\frac{1}{2}}(ilb)\chi'_n = \frac{l'^2}{K_T}\, J_{n+\frac{1}{2}}(k'b)\, \chi''_n \qquad (3)$$

$$\overset{(\circ)}{_p}C_{p,n,1} \left\{ nJ_{n+\frac{1}{2}}(lb) - lb\, J_{n-\frac{1}{2}}(lb) \right\} \phi_n + \left\{ nK_{n+\frac{1}{2}}(ilb) \right.$$

$$\left. + lbi\, K_{n-\frac{1}{2}}(ilb) \right\} \phi'_n$$

$$= \frac{1}{K_T} \left\{ nJ_{n+\frac{1}{2}}(k'b) - k'b\, J_{n-\frac{1}{2}}(k'b) \right\} \phi''_n \qquad (4)$$

The conditions (2) and (3) cannot in general be satisfied, unless

$$\chi_n' = \chi_n'' = 0.$$

Solving (1) and (4) for ϕ_n', ϕ_n'', we get

$$\phi_n' = \overset{(1)}{E_n}\, r'^n P_n(\mu'),$$

$$\phi_n'' = \overset{(1)}{F_n}\, r'^n P_n(\mu),$$

where $\overset{(1)}{E_n}$ and $\overset{(1)}{F_n}$ are two constants.

Thus we get

$$a_1 = \sum_{n=0}^{n=\infty} \overset{(1)}{E_n}\, \frac{K_{n+\frac{1}{2}}(ikr')}{\sqrt{r'}} \frac{1}{}\, P_n(\mu') \sin \phi\, e^{ikVt},$$

etc.,

$$f_1 = \frac{1}{4\pi} \sum_{n=0}^{n=\infty} \frac{\overset{(1)}{E_n}}{2n+1} \left[\frac{K_{n+\frac{3}{2}}(ikr')}{\sqrt{r'}} \frac{1}{} P_{n+1}(\mu') \cos \phi + (n+1)\frac{K_{n-\frac{1}{2}}(ikr')}{\sqrt{}} \right.$$

$$\left. \frac{1}{} P_{n-1}(\mu') \cos \phi \right] e^{ikVt}$$

etc.

$$a'_1 = \sum_{n=0}^{n=\infty} \overset{(1)}{F_n}\, \frac{J_{n+\frac{1}{2}}(k'r')}{\sqrt{r'}} \frac{1}{} P_n(\mu') \sin \phi\, e^{ikVt},$$

etc.,

$$f_1 = \frac{1}{4\pi i r} \sum_{n=0}^{n=\infty} \overset{(1)}{F_n} \frac{1}{2n+1} \left[n\, \frac{J_{n+\frac{3}{2}}(k'r')}{\sqrt{r'}} \frac{1}{} P_{n+1}(\mu') \cos \phi \right.$$

$$\left. - (n+1)\, \frac{J_{n-\frac{1}{2}}(k'r')}{\sqrt{r'}} \frac{1}{} P_{n-1}(\mu') \cos \phi \right] e^{ikVt},$$

Here again, both the reflected disturbance and the disturbance within the sphere B are all of the same type.

The wave system determined by (a_1, b_1, c_1) and (f_1, g_1, h_1) will undergo a second reflection on the surface of the sphere A and so on. We can easily complete the solution.

§ 11.

I now take up the case when the shell A and the conducting sphere B are external to each other. Suppose that a distribution of electricity whose surface density is proportional to the zonal harmonic $P_p (\mu)$ is produced on the surface of the sphere A and that the cause producing this distribution is suddenly removed. Then electrical oscillations will be started on the surface of the sphere A which will travel outwards.

The components of magnetic induction in those waves are given by

$$a_0 = B_p^{(0)} \frac{K_{p+\frac{1}{2}}(ikr)}{\sqrt{r}} P_p^1 (\mu) \sin \phi\, e^{ikVt},$$

$$b_0 = -B_p^{(0)} \frac{K_{p+\frac{1}{2}}(ikr)}{\sqrt{r}} P_p^1 (\mu) \cos \phi\, e^{ikVt}$$

$$c_0 = 0,$$

and the components of polarization are given by

$$f_0 = \frac{B_p^{(0)}}{4\pi(2p+1)} \left[p \frac{K_{p+\frac{3}{2}}(ikr)}{\sqrt{r}} P_{p+1}^1 (\mu) \cos \phi \right.$$
$$\left. + (p+1)\frac{K_{p-\frac{1}{2}}(ikr)}{\sqrt{r}} P_{p-1}^1 (\mu) \cos \phi \right] e^{ikVt}$$

$$g_0 = \frac{B_p^{(0)}}{4\pi(2p+1)} \left[p \frac{K_{p+\frac{3}{2}}(ikr)}{\sqrt{r}} P_{p+1}^1 (\mu) \sin \phi \right.$$
$$\left. + (p+1)\frac{K_{p-\frac{1}{2}}(ikr)}{\sqrt{r}} P_{p-1}^1 (\mu) \sin \phi \right] e^{ikVt}$$

$$h_0 = \frac{B_p^{(0)}p(p+1)}{4\pi(2p+1)} \left[\frac{K_{p+\frac{3}{2}}(ikr)}{\sqrt{r}} P_{p+1} (\mu) - \frac{K_{p-\frac{1}{2}}(ikr)}{\sqrt{r}} \cdot \right.$$
$$\left. P_{p-1} (\mu) \right] e^{ikVt}$$

where $B_p^{(0)}$ is a constant and k is a root of the equation

$$\frac{d}{da} \left[a^{\frac{1}{2}} K_{p+\frac{1}{2}} (ika) \right] = 0.$$

A portion of this disturbance will be incident on the surface of the sphere B and will be reflected.

Proceeding as in §§ 5, 6 and using theorem (IV) given in § 2, we easily find the following expressions for the components (a_1, b_1, c_1) and (f_1, g_1, h_1) respectively in the reflected disturbance :—

$$a_1 = \sum_{n=0}^{} B_n \frac{{}^{(1)}K_{n+\frac{3}{2}}(ikr')}{\sqrt{r'}} P_n^1(\mu') \sin\phi\, e$$

$$b_1 = -\sum_{n=0}^{n=\infty} B_n \frac{{}^{(1)}K_{n+\frac{3}{2}}(ikr')}{\sqrt{r'}} P_n^1(\mu') \cos\phi\, e^{ikVt}$$

$$c_1 = 0,$$

$$f_1 = \frac{1}{4\pi} \sum_{n=0}^{n=\infty} \frac{{}^{(1)}B_n}{2n+1} \left[n\frac{K_{n+\frac{3}{2}}(ikr')}{\sqrt{r'}} P_{n+1}^1(\mu')\cos \right.$$

$$\left. + (n+1) \frac{K_{n-\frac{1}{2}}(ikr')}{\sqrt{r'}} P_{n-1}^1(\mu')\cos\phi \right] e^{ikVt},$$

$$g_1 = \frac{1}{4\pi} \sum_{n=0}^{n=\infty} \frac{{}^{(1)}B_n}{2n+1} \left[n\frac{K_{n+\frac{3}{2}}(ikr')}{\sqrt{r'}} P_{n+1}^1(\mu')\sin\phi \right.$$

$$\left. + (n+1) \frac{K_{n-\frac{1}{2}}(ikr')}{\sqrt{r'}} P_{n-1}^1(\mu')\sin\phi \right] e^{ikVt},$$

$$h_1 = \frac{1}{4\pi} \sum_{n=0}^{n=\infty} \frac{n(n+1){}^{(1)}B_n}{2n+1} \left[\frac{K_{n+\frac{3}{2}}(ikr')}{\sqrt{r'}} P_{n+1}(\mu') \right.$$

$$\left. \frac{K_{n-\frac{1}{2}}(ikr')}{\sqrt{r'}} P_{n-1}(\mu') \right] e^{ikVt}$$

where $\displaystyle \overset{(1)}{B_n} = -B_p^{(0)} \frac{\dfrac{d}{db}\left[b^{\frac{1}{2}} J_{n+\frac{1}{2}}(kb) \right]}{\dfrac{d}{db}\left[b^{\frac{1}{2}} K_{n+\frac{1}{2}}(ikb) \right]} \frac{2n+1}{2n(n+1)} (-1)^p i^n$

$$\underset{a=0}{\text{Lim.}} \int_{-1}^{1} \phi(a, b, z) P_p^1(z) P_n^1(z)\, dz.$$

Similarly, for the second reflected disturbance, and so on.

In this case if the sphere B is very large in comparison with the sphere A, the second reflected disturbance will practically be negligible and we can conveniently obtain approximate expressions for the solution.

Here again the induced current on the surface of the sphere B is of the same type as the inducing current on the surface of the sphere A.

The periods of oscillations are to the order of approximation $(f_0+f_1, g_0+g_1, h_0+h_1)$ given by k which are the roots of

$$\left| \int_{-1}^{1} \frac{\partial}{\partial r} (r^2 R) \, P_p(\mu) \, d\mu \right|_{r=a} = 0,$$

where

$$r \, R = x \, (f_1+f_0)+y \, (g_1+g_0)+z \, (h_1+h_0).$$

APPENDIX

To prove the theorem

$$\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}}\, P_n^m(\mu)\, {\textstyle{\cos \atop \sin}}\, m\phi = \sum_{p=o}^{p=\infty} C_{n,\,p,\,m}\, \frac{J_{p+\frac{1}{2}}(r')}{\sqrt{r'}}$$

$$P_p^m(\mu')\, {\textstyle{\cos \atop \sin}}\, m\phi,$$

where

$$C_{n,\,p,\,m} = (2p+1)\frac{(p-m)!}{(p+m)!}\, i^{\,p-n} \int_{-1}^{1} P_n^m(z)\, P_p^m(z)\, e^{iDz}\, dz.$$

It is easy to see that we have

$$\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}}\, P_n^m(\mu)\, {\textstyle{\cos \atop \sin}}\, m\phi$$

$$= (-1)^{n+\frac{m}{2}}\sqrt{\frac{2}{\pi}}\, \frac{(n+m)\,(n+m-1)\ldots(n+1)}{2\pi}$$

$$\int_0^{2\pi} {\textstyle{\cos \atop \sin}}\, mu\left(\frac{\partial}{\partial z} + i\cos u\,\frac{\partial}{\partial x} + i\sin u\,\frac{\partial}{\partial y}\right)^n \frac{\sin r}{r}\, du.$$

This we write in the form

$$\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}}\, P_n^m(\mu)\, {\textstyle{\cos \atop \sin}}\, m\phi$$

$$= (-1)^n\sqrt{\frac{2}{\pi}}\, S_n^m\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\frac{\sin r}{r}.$$

Now.

$$\frac{\sin r}{r} = \frac{\pi}{\sqrt{Dr'}} \sum_{s=0}^{s=\infty} (-1)^s (s+\tfrac{1}{2})\, J_{s+\frac{1}{2}}(D)\, J_{s+\frac{1}{2}}(r')\, P_s(\mu').$$

Therefore

$$\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{r}}\, P_n^m(\mu)\, \frac{\cos}{\sin}\, m\phi$$

$$=(-1)^n \sqrt{\frac{2\pi}{D}} \sum_{s=0}^{s=\infty} (-1)^s (s+\tfrac{1}{2})\, J_{s+\frac{1}{2}}(D)$$

$$S_n^m\left(\frac{\partial}{\partial x},\, \frac{\partial}{\partial y}\cdot\frac{\partial}{\partial z}\right)\left[\frac{J_{s+\frac{1}{2}}(r')}{\sqrt{r'}}\, P_s(\mu')\right]$$

$$=(-1)^n \sqrt{\frac{2\pi}{D}} \sqrt{\frac{2}{\pi}} \sum_{s=0}^{s=\infty} i^s (s+\tfrac{1}{2})\, J_{s+\frac{1}{2}}(D)$$

$$P_s\left(\frac{\partial}{i\,\partial z'}\right) S_n^m\left(\frac{\partial}{\partial x''},\, \frac{\partial}{\partial y''},\, \frac{\partial}{\partial z'}\right)\frac{\sin r'}{r'}$$

$$= \sqrt{\frac{2\pi}{D}} \sum_{s=\infty}^{s=\infty} i^s (s+\tfrac{1}{2})\, J_{s+\frac{1}{2}}(D)\, P_s\left(\frac{\partial}{i\,\partial z'}\right)\left[\frac{J_{n+\frac{1}{2}}(r')}{\sqrt{r'}}\right.$$

$$\left. P_n^m(\mu')\, \frac{\cos}{\sin}\, m\phi\right].$$

If $\phi(r', \mu', m)$ is a particular integral of the differential equation

$$i^{-m} P_m^m\left(\frac{\partial}{i\,\partial z'}\right)\phi(r', \mu', m) = \frac{J_{m+\frac{1}{2}}(r')}{\sqrt{r'}}\, P_m^m(\mu')$$

which is the same as

$$i^{-m}\left(1+\frac{\partial^2}{\partial z^2}\right)^{\frac{m}{2}}\phi(r', \mu', m) = \frac{J_{m+\frac{1}{2}}(r')}{\sqrt{r'}}(1-\mu'^2)^{\frac{m}{2}}$$

then it is easy to prove by induction that we have

$$i^{-n} P_n^m\left(\frac{\partial}{i\,\partial z'}\right)\phi(r', \mu', m) = \frac{J_{n+\frac{1}{2}}(r')}{\sqrt{r'}}\, P_n^m(\mu').$$

Thus we get

$$P_s \left(\frac{\partial}{i \partial z'} \right) \left[\frac{J_{n+\frac{1}{2}}(r')}{\sqrt{r'}} \; P_n^m (\mu') \right]$$

$$= i^{-n} P_s \left(\frac{\partial}{i \partial z'} \right) P_n^m \left(\frac{\partial}{i \partial z'} \right) \phi (r', \mu', m)$$

$$= i^{-n} \sum E_{n, \, p, \, m} \; P_p^m \left(\frac{\partial}{i \partial z'} \right) \phi (r', \mu', m)$$

$$= i^{-n} \sum_p i^p \, E_{n, \, p, \, m} \; \frac{J_{p+\frac{1}{2}}(r')}{\sqrt{r'}} \; P_p^m (\mu'),$$

where

$$E_{n, \, p, \, m} = \frac{2p+1}{2} \; \frac{(p-m)!}{(p+m)!} \int_{-1}^{1} P_s (z) \, P_n^m (z) \, P_p^m (z) \, dz.$$

Therefore

$$\frac{J_{n+\frac{1}{2}}(r)}{\sqrt{}} \, P_n^m (\mu) \; \genfrac{}{}{0pt}{}{\cos}{\sin} \, m \, \phi = \sum_{p=0}^{p=\infty} C_{n, \, p, \, m} \; \frac{J_{p+\frac{1}{2}}(r')}{\sqrt{}}$$

$$P_p^m (\mu') \; \genfrac{}{}{0pt}{}{\cos}{\sin} \, m \, \phi.$$

where

$$C_{n, \, p, \, m} = \sqrt{\frac{2\pi}{D}} \; i^{p-n} \, \frac{2p+1}{2} \frac{(p-m)!}{(p+m)!} \int_{-1}^{1} P_n^m (z)$$

$$P_p^m (z) \left[\sum_{s=0}^{s=\infty} i^s \, (s+\tfrac{1}{2}) \, J_{s+\frac{1}{2}} (D) \, P_s (z) \right]$$

$$= i^{p-n} \, (2p+1) \, \frac{(p-m)!}{(p+m)!} \int_{-1}^{1} P_n^m (z) \, P_p^m (z) \, e^{iDz} \, dz$$

Theorems (IV) and (V) can be proved in a similar manner if we notice that

$$
\frac{K_{n+\frac{1}{2}}(r)}{\sqrt{r}} \; P_n^m(\mu) \; {\cos \atop \sin} \; m\phi
$$

$$
= \sqrt{\frac{\pi}{2}} \; i^{n+1} \sqrt{i} \; \frac{(n+m)(n+m-1)\cdots(n+1)}{2\pi} \; (-1)^{\frac{m}{2}}
$$

$$
\int_0^{2\pi} {\cos \atop \sin} \; mu \left(\frac{\partial}{\partial z} + i\cos u \frac{\partial}{\partial x} + i\sin u \frac{\partial}{\partial y} \right) \frac{e^{-ir}}{r} \; du.
$$

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Illustrating Large-Angle Diffraction

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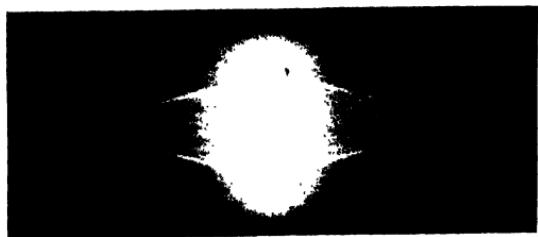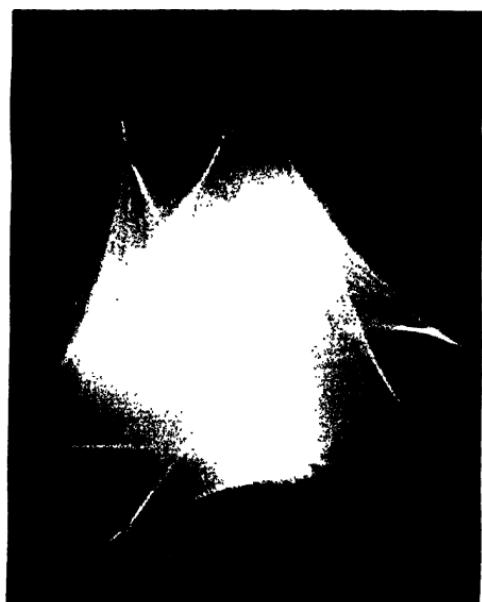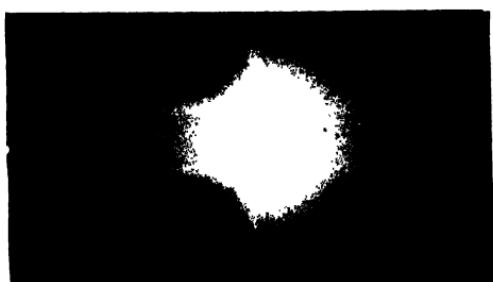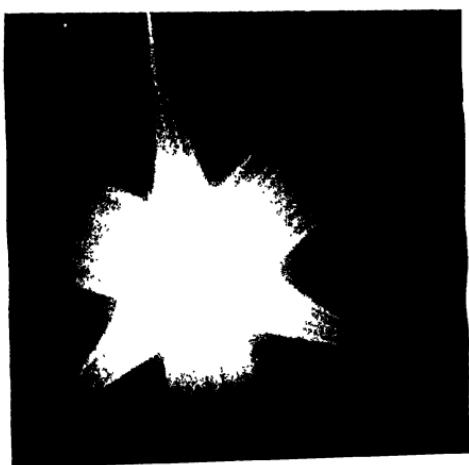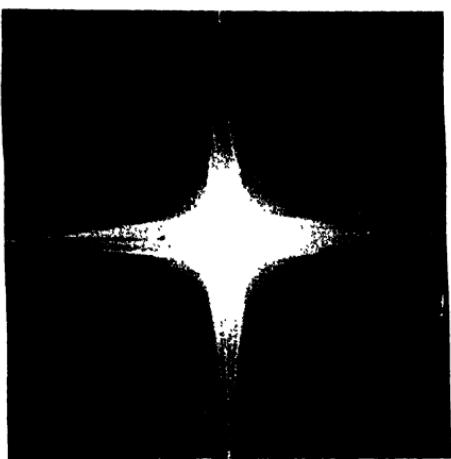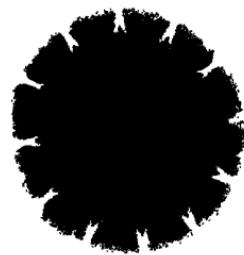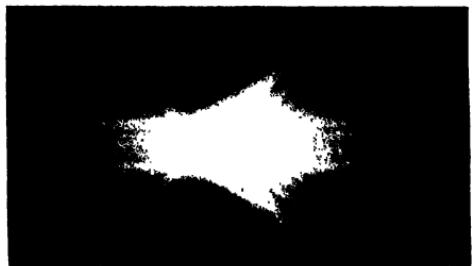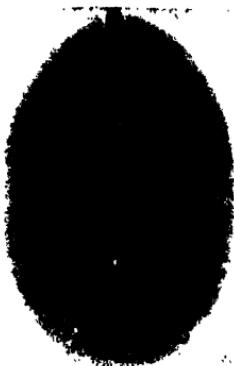
On Some Studies of Large-Angle Diffraction.

BY

Sisirkumar Mitra, M.Sc.

1. · *Introduction.*

The scattering of light by a rectilinear diffracting edge has been studied by various experimenters among whom may be mentioned principally Gouy,[1] Wien,[2] E. Maye[3] and more recently also Kalaschnikow[4] who has used a photographic method. The rigorous theory for the case was given by Sommerfeld[5] in the famous memoir on the Mathematical Theory of Diffraction in which he dealt with the case of a semi-infinite screen and developed a general expression for the intensity applicable both for small and large angles of diffraction. The analogous phenomenon of the large angle diffraction by a curvilinear boundary does not however appear from the literature of the subject to have received much attention. Experimenters as for instance, W. B. Croft[6] and more recently also Arkadief[7] and Gordon[8] who studied the diffraction phenomena of the Fresnel class due to various forms of apertures have practically confined their attention to the parts of the diffraction field for which the angle of deviation is small, and in fact the photographs reproduced with their papers show only the areas of the field which are relatively intensely illuminated and lie within, or very near the geometrical region of transmission of the rays passing through the aperture. It cannot be claimed that the results obtained by these workers give an adequate picture of the phenomena of diffraction by apertures of the forms considered. Judging from the analogy with the case of the semi-infinite screen treated in Sommerfeld's paper,

[1] Gouy, Ann. d. Phys. et. de. chemi, (6), 8, P. 145 (1886).
[2] W. Wien, Wied. Ann. 28, P. 117 (1886).
[3] E. Maye, Wied Ann. 49, P. 69 (1893).
[4] Kalaschinikow, Journ. Russ. Phys, Ges, 44 (1912).
[5] Sommerfeld, Math. Ann. t XLVII, P. 317 (1895).
[6] W. B. Croft, Phil. Mag. S 5, V. 38 (1894).
[7] Arkadief, Phys. Zeit, 14 (1913).
[8] Gordon, Pooc. of Lond. Physical Society, 24 Oct. (1912).

the intensity of the diffracted waves should be sensible over an area enormously greater than that of the cross-section of the geometrical pencil of rays passing through the aperture, and we should expect the distribution of luminosity over such large areas to depend on the precise form of the aperture used in a much more striking manner than is possible within the limits of the geometrical region of transmission. If the analogy could be stretched a little further, it might indeed be said that we have to proceed outside the limits of this region in order that the true " waves of diffraction" having their origin at the boundaries of the aperture might be separately observed. An investigation of the distribution of luminosity in the fainter outlying regions in diffraction-pattern of the Fresnel class is thus evidently of considerable interest and importance, and the present work was undertaken with a view to studying a few typical cases of the kind, and thus ascertaining the general features of the phenomenon.

In order that the best results might be obtained, it is necessary that the edges of the apertures used should be smooth and highly polished. The cases for which the large angle diffraction have been studied in the present investigation are the following :—

(a) Elliptic apertures, (b) semi-circular apertures and other forms of aperture bounded by arcs of circles and straight lines, (c) apertures or screens with undulating or corrugated boundaries. The experimental method adopted was to use a brilliantly illuminated point source, the light from which after passing through the diffracting aperture falls upon a photographic plate held at a suitable distance from it. By using prolonged exposures, the fainter regions of luminosity surrounding the Fresnel pattern may be recorded on the plate, which is then studied at leisure. The brighter parts of the field of course become greatly over-exposed and cease to show any detail. A few selected photographs from those obtained are reproduced in the Plate, Figs. (a to l).

2. *Elliptic Apertures.*

The case of the elliptic aperture was the first to be studied. If the eccentricity of the ellipse be small, nothing of special interest is noticeable outside the limits of the geometrical region of transmission. But as the eccentricity is gradually increased till the ratio of the major to the minor axis exceeds $\sqrt{2}$, some interesting features

develop which become more and more striking as the ratio is increased. Fig (*a*) in the Plate shows the effect. It will be noticed that the luminosity is particularly marked along four curved arcs which measurements show to coincide exactly with the evolute of the elliptic boundary in which the plane of observation intersects the transmitted pencil of rays. Further, on closer inspection it is seen that running parallel to these arcs, on the inner side a few alternately dark and bright fringes appear, which are somewhat analogous to the interference fringes seen alongside a caustic. The luminosity becomes fainter and fainter as we recede from the centre of the field, but it does not terminate at the cusps of the evolute, continuing beyond it in the form of two horizontal brushes, which extend one on each side. (The photographic plate was not large enough to cover this part of the field). Outside the region enclosed by the evolutes, luminous streamers may also be seen radiating normally from all parts of the elliptic boundary.

The phenomena described above are always noticed whether the incident pencil of rays is divergent or convergent. But interesting effect is observed in the case of convergent light as the plane of observation approaches the focal plane. The cross-section of the pencil and the luminous arcs of the evolute appearing on either side of it gradually contract and reduce to a point. The horizontal brushes extending outwards from the cusps however persist. These break up into beads of light on both sides, which when the focal plane is further approached ultimately join up and form the elliptic rings usually observed in the Fraunhofer pattern at the focus.

Fig. (*g*) in the plate shows the faint region of luminosity inside the shadow of an elliptic disc. The four luminous arcs forming the evolute of the boundary are clearly noticeable.

3. *The case of a semi-circular Aperture.*

This case is an instructive example of the importance of studying the phenomena of large-angle diffraction. If we exclude the outlying regions of faint luminosity, the diffraction pattern of the Fresnel type is an approximately semi-circular patch of light, with fringes running inside it parallel to the bounderies, and in fact this is how it appears in a photograph reproduced in the paper by Gordon quoted above. In this form, the Fresnel pattern bears not the remotest resemblance to

the diffraction figure of the Fraunhofer class due to a semi-circular aperture as observed at the focus of a Heliometer objective. A study of the large angle diffraction however puts the case in an entirely different aspect. Fig. (e) in the Plate shows the effects observed. It will be noticeable that there is a long horizontal trail of light proceeding to a great distance in either direction at right angles to the diameter of the semi-circle. Crossing this, we have numerous streamers of light, one of which is parallel to the diameter and the others form roughly a system of hyperbolic arcs symmetrical about the horizontal axis.

The streamers are brightest to the left of the diameter, but they may also be seen very faintly to the right of it. The pattern thus shows many striking resemblance with the Fraunhofer pattern at the focus of the heliometer objective which is described by Everitt in the following words.

"The image is symmetrical about two perpendicular axes. There is a central approximtely oval spot which is roughly a little less than twice as long as it is broad. This central spot is bisected along the major axis by a long bright ray which fades away as it recedes from the centre. Around the spot and following its outline are a series of alternate bright and dark rings, which at the minor axis, are identical with those of the diffraction image of a circular aperture, and along the major axis, terminate at an angle in a series of bright spots. Further, these rings are crossed and modified by a series of alternate bright and dark rays of approximately hyperbolic shape, having the major axis of the figure for their geometric axis, which they cut at its maxima and minima, with the exception that the first one of the series passes through a point in the central bright spot."

Fig. (j) in the Plate represents the large angle diffraction pattern due to a circular segment smaller than a semi-circle and is instructive in the comparison it affords with the case just considered. The two cases have many features in common. The transverse streamers obtained in the case of the segment are however much clearer and cover a smaller part of the field than in the case of the semi-circle. On one side they may be clearly seen radiating from a point which lies outside the strongly illuminated part of the field and is in fact the centre of the circular boundary.

9. P. F. Everitt—Proceedings of the Royal Society Vol. 83 (1910).

Figs. (*l*) and (*k*) in the Plate show the large angle diffraction due to apertures bounded by three and four arcs of circles respectively the centres of which lie outside the area of the apertures. The streamers of light starting out normally from the projection of the boundaries are clearly visible in the photographs. On one side of each they diverge and on the other side they converge to foci at the respective centres and then diverge again. The streamers are not all equally bright. The fluctuations of intensity near the margin of each group being particularly noticeable.

4. · *Plane aperatures or screens undulating or corrugated boundaries.*

The case of apertures with undulating boundaries are of particular interest. The projection of the boundary of the aperture on the plane of observation being also undulating, its evolute has one asymptote for each point of inflexion of the boundary, and a cusp for each point at which the curvature of the boundary passes through a maximum value. Broadly speaking, the general from of the large-angle diffraction pattern is determined by the form of the evolute. This is illustrated by Fig. (b) in the Plate, which refers to ι case in which the aperture was bounded by three arcs, each of which contained a point of inflexion. The three asymptotic lines limiting the from of the pattern can clearly be made out.

Figs. (*d*) and (*h*) in the Plate show the distribution of luminosity within the shadow of a disc with 12 undulations in its margin (this was an one-anna nickel coin). The concentration of luminosity along the evolute (with 12 cusps) of the geometrical boundary of the shadow is clearly seen. Parallel to the evolute run a series of alternately ·dark and bright fringes of decreasing luminosity, and radial brushes of light also emerge from the cusps of the evolute.

Fig. (*c*) shows the distributions of luminosity inside the shadow of a square screen with rounded corners.

When the undulations are so very numerous and close together as to form corrugations of very small radius of curvature, *interesting new phenomena arise.* Fig. (*f*) shows the observed distribution of illumination within the shadow of a circular disc having a milled edge. The total number of corrugations on the circumference was 162 and the diameter of the disc was 2·94 cm. In addition to a central spot

analogous to that observed in the well-known Fresnel-Arago circular
disc experiment, we have a circular ring rather brighter than the
central spot at a considerable angular distance from it, and one or
two other fainter rings further· outside. With white light these rings
were found to be sharply defined circular spectra somewhat similar
to those produced by a circular grating, and the central spot was
also coloured in certain cases. These rings are due to the corrugation
on the edge of the disc and are situtated at much greater distances
from the central spot and from each other than the Lommel rings
round the central spot of the shadow on a circular disc. The relative
intensities of the central spot and rings were bound to depend
upon the exact form of the corrugations on the edge. The explana-
tion of these phenomena will be referred to again later.

In order to further illustrate the large-angle diffraction
by a plane corrugated boundary, the case of a screen bounded on
one side by a circular arc with a corrugated edge was studied. Fig.
(*i*) shows the pattern produced by such an obstacle. Streams of
light radiating from the corrugations can be seen to have come to
a focus at the centre of the arc and also to other foci lying on either
side of it. Each one these foci in white light is a little spectrum,
and their relative intensities depend on the exact form of the corru-
gations on the edge.

Theory.

The experimental results described above suggest that the large-
angle diffraction is practically a boundary effect. According to
Kirchhoff's formulation of Huyghen's principle as applied to the expla-
nation of diffraction phenomena, the effect at any point of the wave
passing through the aperture is expressed as a surface integral taken
over the area of the aperture. Thus, in order to show how the value
of the diffraction integral at such point depends on the form of the
boundary of the aperture, we have to transform the surface-integral
into a line-integral taken over the boundary. This may be done
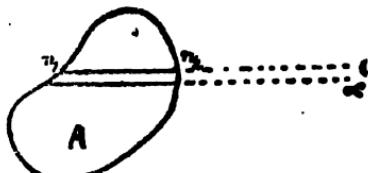
in the following manner. Let A · represent the aperture (Fig. 1)

Fig. I

cut in an infinite opaque screen. We may divide the aperture into a large number of parallel strips, n_1 n_2 etc. The effect of n_1 n_2 is obviously equivalent to that of a strip starting from n_1 and extending up to infinity less that due to a strip extending from n_2 to infinity. To calculate the effect of one of these strips say $(n_1 \quad \infty)$ assume a set of rectangular coordinates (Fig. 2) such that the plane of the aperture A (cut in a plane screen of infinite extent) is in the XY plane, and the projection of SP (S being the source and P the point at which the effect is sought) on the plane of the aperture is the X axis, the origin lying on SP and in the plane of the screen.

Fig. 2
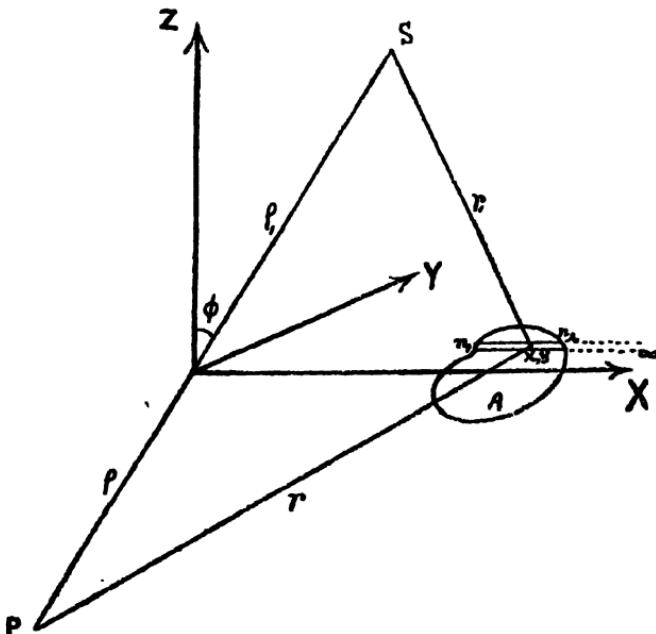
Let ρ_1 and ρ be the distances of S and P from the origin; (x, y) the coordinates of an elementary portion the strip (which is taken parallel to the X axis), r_1 and r the distances of S and P respectively from (x, y) and x', y' the coordinates of n_1. The effect S_0 at P due to the whole strip may be written

$$S_0 = \frac{A_0}{\lambda} \int_{x'}^{\infty} \frac{\cos(n\,r) - \cos(n\,r_1)}{r r_1} \sin 2\pi \left(\frac{t}{\tau} - \frac{r + r_1}{\lambda} \right) d\sigma$$

where. A_o is the amplitude of light disturbance at unit distance from S, (nr) and (nr_1) are the angles which the normal at (x, y) makes with r and r_1 respectively. In this equation we can take the quantities $r \, r_1$ as constant whenever they are not divided by λ because a limited region of integration near n_1 is determinitive of the intensity at P, this region including the most powerful zones (and a large number of them) and an extension of the integration over a larger region adds nothing to the intensities. As an approximation we may also in making the integration take $\cos(nr) - \cos(nr_1) = $ a constant $(2K)$ for the particular point at which the effect is required. Therefore

$$S_o = \frac{A_o}{\lambda r_1} \frac{K}{r} \int \sin 2\pi \left(\frac{t}{\tau} - \frac{r+r_1}{\lambda} \right) d\sigma$$

If ϕ is the angle made by ρ with z axis we may write

$$r_1 + r = \rho_1 + \rho + \tfrac{1}{2} \left(\frac{1}{\rho_1} + \frac{1}{\rho} \right) \left[x^2 \cos^2\phi + y^2 \right]$$

Changing the origin of time in the equation of S_o by writing

$$\frac{t'}{\tau} = \frac{t}{\tau} - \frac{\rho_1 + \rho}{\lambda} - \frac{1}{2\lambda} \left(\frac{1}{\rho_1} + \frac{1}{\rho} \right) y^2$$

<div align="right">(since y is const; the strip being
parallel to the X axis.)</div>

S_o might be written to be equal to

$$A'dy \int \sin 2\pi \left(\frac{t'}{\tau} - \frac{1}{2\lambda} \left[\frac{1}{\rho_1} + \frac{1}{\rho} \right] x^2 \cos^2\phi \right) dx$$

where $A' = \dfrac{A_o K}{\lambda r r_1}$ and $d\sigma = dx \, dy$, dy being the width of the strip.

If $F(x) = \dfrac{\pi}{\lambda} \left(\dfrac{1}{\rho_1} + \dfrac{1}{\rho} \right) x^2 \cos^2\phi$

$$S_o = A' \, dy \left\{ \sin 2\pi \frac{t'}{\tau} \int \cos F(x) \, dx + \cos 2\pi \frac{t'}{\tau} \int \sin F(x) \, dx \right\}$$

Let $C = \displaystyle\int_{x'}^{\infty} \cos F(x) \, dx$ and $D = \displaystyle\int_{x'}^{\infty} \sin F(x) \, dx$

Then the amplitude of the resultant vibration is given by

$$S_o = A'dy\sqrt{C^2+D^2}$$

and the phase α, by $\tan \alpha = \dfrac{D}{C}$

To evaluate C and D

put $\dfrac{\pi}{\lambda}\left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho}\right)s^2\cos^2\phi = \dfrac{\pi}{2}v^2$

then $dv = \sqrt{\dfrac{2}{\lambda}\left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho}\right)}\cos\phi\, ds$

Therefore $S_o = f'dy\left\{\sin 2\pi\dfrac{l'}{\tau}\displaystyle\int_{v'}^{\infty}\cos\dfrac{\pi}{2}v^2\,dv+\cos 2\pi\dfrac{l'}{\tau}\displaystyle\int_{v'}^{\infty}\sin\dfrac{\pi}{2}v^2\,dv\right\}$

where $f' = \sqrt{\dfrac{2}{\lambda}\left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho}\right)}\cos\phi$ and $v' = \sqrt{\dfrac{2}{\lambda}\left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho}\right)}s'\cos\phi$

By using the semi-convergent expansions we have

$$\int_{v'}\cos\dfrac{\pi}{2}v^2\,dv = M\cos\dfrac{\pi}{2}v'^2 - N\sin\dfrac{\pi}{2}v'^2$$

$$\int_{v'}^{\infty}\sin\dfrac{\pi}{2}v^2\,dv = M\sin\dfrac{\pi}{2}v'^2 + N\cos\dfrac{\pi}{2}v'^2$$

Where $M = \dfrac{1}{\pi^2 v'^3} - \dfrac{1.3.5}{\pi^4 v'^7} + \dfrac{1.3.5.7.9}{\pi^6 v'^{11}} + $ etc.

and $N = \dfrac{1}{\pi v'} - \dfrac{1.3}{\pi^3 v'^2} + \dfrac{1.3.5.7}{\pi^5 v'^5} + $ etc.

As we are dealing with large angle diffraction v' is large and it is sufficient to retain the first term only and we may write

$$\int_{v'}^{\infty}\cos\dfrac{\pi}{2}v'^2\,dv = \dfrac{1}{\pi^2 v'^3}\cos\dfrac{\pi}{2}v'^2 - \dfrac{1}{\pi v'}\sin\dfrac{\pi}{2}v'^2$$

$$\int_{v'}^{\infty}\sin\dfrac{\pi}{2}v'^2\,dv = \dfrac{1}{\pi^2 v'^3}\sin\dfrac{\pi}{2}v'^2 + \dfrac{1}{\pi v'}\cos\dfrac{\pi}{2}$$

Consequently the resultant amplitude at P

$$S_o = f' \, dy \, \sqrt{\left(\frac{1}{\pi^2 v'^3}\right)^2 + \left(\frac{1}{\pi v'}\right)^2}$$

Neglecting $\left(\dfrac{1}{\pi^2 v'^3}\right)^2$ which is very small compared

with $\left(\dfrac{1}{\pi v'}\right)^2$ the amplitude is equal to

$$\frac{f' dy}{\pi v'} = \frac{A_o \, k \, dy}{2\pi \, (\rho_1 + \rho) x' \, \cos^2 \phi}$$

The phase a is given by

$$\text{Cot } a = - \frac{M \cos \frac{\pi}{2} v'^2 - N \sin \frac{\pi}{2} v'^2}{M \sin \frac{\pi}{2} v'^2 + N \cos \frac{\pi}{2} v'^2} = \cot\left(\frac{\pi}{2} v'^2 + \beta\right)$$

where $M = \dfrac{1}{\pi^2 v'^3} = \cos\beta$

$\qquad$ and $\cot\beta = \dfrac{M}{v'^2}$

$\qquad\quad N = \dfrac{1}{\pi v'} = \sin\beta$

Since v' is very large compared to 1, $\dfrac{M}{N}$ is very small and the angle β is very approximately equal to $90°$. Since $\dfrac{\pi}{2} v'^2$ is the phase of the vibration sent out from (x', y'), we may say that the phase of the resultant vibration due to the whole of the strip at P. is the same as the phase of the vibration sent out by n_1 the head of the strip plus $\dfrac{\pi}{2}$.

The resultant vibration due to the whole strip of width dy

$$= \frac{A_c \, K \, \cos\theta}{2\pi \, (\rho_1 + \rho) \, x' \cos^2\phi} \, \sin 2\pi\left(\frac{t}{\tau} - \frac{r + r_1}{\lambda} - \frac{1}{4}\right) ds$$

where θ is the angle which the normal to the element of the edge ds makes with the X axis, so that $dy = ds \cos\theta$. Similarly

the effect of the strip extending from n_2 to ∞ might be represented by a vibration sent out from the elementary boundary at n_2. So that each strip acts as if only its two extremities lying on the boundary of the aperture send out waves to P of definite amplitudes (depending on the length, distance, and the inclination to the X axis of the element of the boundary) and of phases equal to $\psi + 90^\circ$ and $\psi - 90^\circ$, ψ and ψ' being the phases of the vibrations from the two extremities. Thus if the whole aperture be divided into parallel strips the effect produced by the whole aperture might be regarded as the superposition of waves sent out by pairs of elementary portions of the boundary, the phase of the two vibrations in each pair differing by 180.$^\circ$

In the case of an obstacle we may obviously proceed in the same way to find the illumination at any point situated inside the geo-

Fig. 3
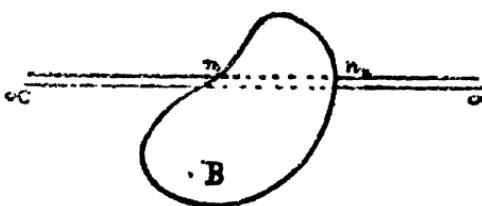
metrical shadow. Thus if in the diagram (Fig. 3) B be an opaque screen, the total effect is due to the sum of strips like ($n_1 \infty$) and ($n_2 \infty$) and the effect of each of these strips as before might be represented by sets of waves sent out from the extremities n_1 and n_2 only.

We may proceed now to consider the explanation of the phenomena illustrated in the Plate. These are ; (a) the luminosity observed along the evolutes of the boundary of the shadow when the edge is smooth or undulating and not highly corrugated, (b) the dark and bright fringes running nearly parallel to the evolute, (c) the special phenomena observed whenever the edge is highly corrugated.

(a) Any point on the evolute is situated at the centre of curvature of some part of the boundary of the shadow of the aperture. By simple geometry it follows that waves sent out from three contiguous points at the corresponding part of the boundary of the aperture

will reach that particular point on the evolute in the same phase
and will conspire to raise the amplitude there considerably above
the amplitude at neighbouring points on either side of the evolute.
Consequently the evolute itself of the boundary will be practically
a region of maximum illumination.

(b) Consider the case of a point situated near any point on
the evolute on its convex side. A little consideration will show
that it is impossible from such point to draw any normal to the
shadow of the boundary, (at any rate near to that particular region
of the boundary which corresponds to the point on the evolute).
Consequently in the region on the concave side no two waves from
the boundary can arrive in the same phase, and the region would
receive very little illumination (unless of course a normal from some
other portion of the boundary passes through it). If the point be
situated on the convex side, it will be seen (vide Fig. 4. where B
is the boundary, E its evolute, and P the point) that it would be

<p align="center">**Fig. 4**</p>
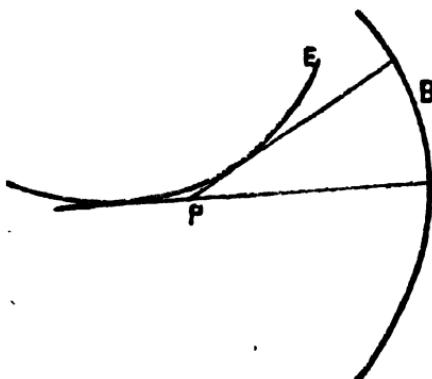
possible to draw two normals through P and since there would be
a difference of path between the two waves arriving at P, they could
interfere and produce a maximum or minimum of illumination.
This explains the fluctuation of insensity seen alongside the evolute.
As P recedes farther and farther from the evolute the two contiguous
points which send the waves to P become more and more distant
from one another, and the illumination due to either becomes insig-
nificantly small.

(c) When the undulations on the edge are numerous and close together, the evolute becomes a highly complicated curve and the phenomena assume a different character owing to the superposition of the effects of different parts of the boundary. Each of the corrugations on the boundary practically acts as a source of radiation, (the intensity of the radiation in any direction depending on the exact from of the indentation) and the phenomena observed within the region of shadow are due to the interference of these effects. Within the particular disc used, the first ring was brighter than the central spot (Plate, Fig. κ), but this is not alway the case. The distance between the corrugations being ·057 cm., the distance of the plane of observation from the disc 350 cm, and the wave-length of the light employed being 4200 A. U., the radii of the successive rings are easily found on the above hypotheses to be ·26 cm,·52 cm, etc. The value obtained by measurements on the photographic plate are ·27 cm, ·57 cm. A fuller treatment of the radiation from a corrugated edge would be very interesting in virtue of the analogy with the problem of the influence of the groove-form on the spectral intensity in the theory of gratings.

The experimental work forming the subject of this paper was carried out in the Palit Laboratory of Physics. The author's best thanks are due to Prof. C. V. Raman who made some preliminary observations on the large-angle diffraction by an elliptic aperture and at whose suggestion the present investigation was undertaken.

On the Colours of the Striæ in Mica

By Prof. C. V. Raman and Mr. Phanindranath Ghosh, with a Note by the Rt. Hon. Lord Rayleigh, O.M., F.R.S.[*]

On examining even the most regularly split and transparent pieces of mica by diffuse reflected light, a few fine hair-like and rather irregular lines may generally be seen running along the surface. We have found that these lines or striæ show some very interesting effects when mica is examined in a Topler "Schlieren" apparatus. The sheet as a whole, being optically good, remains invisible, but the striæ shine out as brilliant and vividly coloured lines of light, the colours being different for different striæ, and changing in a remarkable manner as the inclination of the mica relatively to the direction of the light in the apparatus is altered. For instance, a striæ at normal incidence may appear crimson and, as the mica is rotated about an axis in its own plane, become successively purple, green, yellowish-green, yellow, orange, scarlet-red, green, yellow, and red.

The phenomenon is being investigated in detail by one of us (P. N. Ghosh), but as to its general nature there appears to be little doubt. The striæ are lines at which the thickness of the mica changes in a discontinuous manner, and the luminosity is due to the radiation from the discontinuity acting as a laminar diffracting boundary. For any particular wave-length the radiation is zero if the retardation of the wave-front on either side of the discontinuity differs by an even multiple of half a wave-length, and is approximately a maximum if the difference is an odd multiple of half a wave-length. The detailed mathematical investigation would follow the general lines indicated by Lord Rayleigh in his theory of the Foucault "knife-edge" test (*Phil. Mag.*, February, 1917).

<div align="right">C. V. Raman,</div>

210, Bowbazar Street, Calcutta, India, September 5.   P. N. Ghosh.

Probably the striæ, regarded by the authors as boundaries between regions of slightly different thicknesses, are the same lines as can be seen by reflections of soda light, as described in my note on " Regularity of Structure in Actual Crystals" (*Phil, Mag.*, vol. xix., p. 96, 1910). Doubtless the Foucault method shows them in a more striking manner, and in any case, the colour effects are novel, so far as I know, and worthy of a closer examination.

<div align="right">Rayleigh.</div>

THE KINEMATICS OF BOWED STRINGS.

BY

C. V. RAMAN, M.A.

1. *Introduction.*

It has long been known that if a string be bowed *exactly* at one of its points of aliquot division, the harmonics having a node at that point fail to be elicited, but that if the bow be moved to a point only slightly distant from the node and applied with suitable pressure, the same harmonics are elicited with great vigour, and in certain circumstances may even transcend the fundamental component of the vibration in their intensity. This fact attracted the attention of experimenters early in the history of the subject and naturally assumes prominence in any exposition of the theory of bowed strings. Among the more recent workers who have studied the phenomenon may be mentioned Krigar-Menzel and Raps,[1] and also Davis[2] who noticed a similar effect in the case of the longitudinal vibration of rubbed strings. Neither these experimenters nor later theorists such as Lippich[3] and Andrew Stephenson[4] have however succeeded in clearing up the nature of the transition between the cases in which the partial vibrations fail to appear, and those in which they re-assert themselves with great intensity. The question is of considerable interest in relation to the mechanical theory of the action of the bow, and has (along with various other problems) been discussed by me in the first part of a monograph on bowed stringed instruments which has recently been published.[5] The treatment of the theory given in the monograph is largely of a graphical character, and though the formulæ for the harmonic analysis of the motion were given, they were not fully discussed. It is thought that an exposition of the subject from a more strictly analytical point of view may be of advantage. This is given in the present paper.

[1] Sitzungberichte of the Berlin Academy, 1891.
[2] American Academy of Sciences, Proc., 1906.
[3] Wien Berichte, 1914.
[4] Phil Mag, Jan., 1911.
[5] Bulletin No. 15 of the Indian Association for the Cultivation of Science, Calcutta, 1918.

2. *Analysis of Discontinuous Wave-Motion.*

It is useful to commence by deriving the formulæ which give the amplitude and phase of the harmonics of a vibration involving only discontinuous changes of velocity. The distribution of transverse velocity over the whole string at any epoch may be represented by a diagram (which we shall refer to as the velocity diagram) in much the same way as the configuration of the string at any instant may be represented by a displacement diagram. Since, by assumption, the vibration at every point on the string is determined solely by discontinuous changes of velocity, the velocity diagram of the string must consist of a number of straight lines inclined to the x—axis at the same angle and separated by discontinuities which move to the right or the left according as they belong to the positive or the negative wave, and on reaching an end of the string suffer reflexion and return. The transverse velocity of every point on the string remains unaltered except when a discontinuity passes over it in one direction or the other, when it suddenly alters by a quantity equal to the magnitude of the discontinuity.

Let d_1, d_2, d_3, etc, be the magnitudes of the discontinuities and let their positions on the string at time t be given by $x = c_1, c_2, c_3$ etc. If y be the transverse displacement at any point on the string, it is readily shown by applying the Fourier Analysis to the velocity diagram that

$$\left(\frac{dy}{dt} \right)_t = \sum_1^\infty A_n \sin \frac{n\pi x}{l}, \quad \text{where}$$

$$A_n = -\frac{2}{n\pi} \left[d_1 \cos \frac{n\pi c_1}{l} + d_2 \cos \frac{n\pi c_2}{l} + etc. \right] \qquad \dots \ (1)$$

The quantities c_1, c_2, c_3 etc., are not constant but vary with time. Accordingly, we may write $c_r = (c_r + a\,t)$, or $2l - (c_r + a\,t)$ according as the discontinuity d_r belongs to the positive or the negative wave. On making the necessary substitutions, and separating the sine and cosine components, we get as the equation for the vibration

$$y = \sum_1^\infty \sin \frac{n\pi x}{l} \left[a_n \sin \frac{2n\pi t}{\tau} + b_n \cos \frac{2n\pi t}{\tau} \right]$$

$$\text{where } a_n = -\frac{1}{n^2 \pi \tau} \left[d_1 \cos \frac{n\pi c_1}{l} + d_2 \cos \frac{n\pi c_2}{l} + etc. \right]$$

$$\text{and } b_n = -\frac{1}{n^2 \pi^2 \tau} \left[d_1 \sin \frac{n\pi c_1}{l} + d_2 \sin \frac{n\pi c_2}{l} + etc. \right] \quad \dots \ (2)$$

It will be seen that, generally speaking, both sine and cosine functions of the time are involved in the expression for the displacement, and that it is not possible to get rid of the cosine terms by merely changing the origin of time. In general, therefore, the string does not coincide with its position of equilibrium at any epoch of the vibration. The cosine terms will however vanish provided that the positive and negative waves are of identical form and coincide in position at the epoch chosen as the origin of time. This is readily verified from (?) above, as the co-efficients b_n all vanish, and in the resulting vibration, the string everywhere passes through the position of equilibrium at two epochs in each period of vibration.

Reference may be made here to three papers recently contributed to the Philosophical Magazine in which it has been shown how certain simple discontinuous types of vibration may be experimentally realised.[1]

3. *Vibrations with the complete series of Partials.*

In order that the oscillation should include the complete series of partials, it is necessary to assume that the bowed "point" divides the string in an irrational ratio. If the motion at this point is known, the entire vibration of the string is kinematically determinate. For the purpose of the discussion, it may first be assumed that the bowed point of the string moves to and fro, once or oftener in each period of vibration, with constant and uniform velocities. Mechanical theory indicates that this is the type which the motion at the bowed point approximates to, but does not necessarily attain in any particular case. The velocity of the bowed point when it moves with the bow may be denoted by v_b and when it moves in the opposite direction, v_a. The velocity diagram of the string must be of such form that by the passage of the discontinuities over it, the velocity at the bowed point alternates between v_a and v_b once or oftener in each period of vibration. It is obvious that if the magnitude of all the discontinuities in the velocity diagram is the same and equal to $(v_a - v_b)$, and that two or more discontinuities do *not* pass over

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[1] *On discontinuous wave motion*, Phil. Mag. Jan. 1916, Feb., 1918 and April, 1918.

the bowed point in succession in the *same direction*, the required
type of motion at the bowed point would be secured. For, the
initial velocity at the bowed point being taken to be v_b the
velocity changes to v_a when a discontinuity passes over it in one
direction, and changes back to v_b when the same or another dis-
continuity passes over it in the opposite direction. By pursuing
the argument on these lines, it may be shown that if the bowed
point divides the string in an irrational ratio, and its motion is
strictly of the type contemplated, the magnitudes of the discon-
tinuities in the velocity diagram of the string are necessarily all
equal to $(v_a - v_b)$. We shall here assume this result.

The magnitude of the discontinuities in the velocity-diagram being
taken to be equal to $(v_a - v_b)$, the amplitudes and phases of the
harmonics in the vibration may be readily calculated. The number
of discontinuities in the velocity-diagram gives us a simple criterion
for classification of the modes of vibration :

Case of one discontinuity :

This is the simplest type of all. If the origin of time chosen
be the epoch at which the discontinuity is at the end of the string,
the cosine terms in the expression for the displacement vanish, and
we get

$$y = \sum_{1}^{\infty} \frac{-(v_a - v_b)}{n\pi^2 T} \sin \frac{n\pi x}{l} \sin \frac{2n\pi t}{\tau} \qquad \ldots \qquad \ldots \quad (3)$$

This is the well-known principal type of vibration of a bowed
string discovered by Helmholtz in which the vibration-curve at every
point on the string is a simple two-step zig-zag. The ratio r_a/v_b is
equal to the ratio in which the bowed point divides the string.

Case of two equal discontinuities :

The kinematics of this case is readily worked out in detail. It is
sufficient for our present purpose to give an analytical demonstration
of two important features in regard to this type of vibration. If
the origin of time chosen be the instant at which the two disconti-
nuities coincide in position, the cosine terms vanish, and we get

$$y = \sum_{1}^{\infty} \frac{-2(v_a - v_b)}{n^2 \pi^2 T} \sin \frac{n\pi x}{l} \cos \frac{n\pi c}{l} \sin \frac{2n\pi t}{\tau} \qquad \ldots \quad (4)$$

where c is the distance from the end of the string of the point at which the two discontinuities cross. At the point $x = c$, we have

$$y = \sum_{1}^{\infty} \frac{-(v_a - v_b)}{n^2 \pi^2 T} \sin \frac{2n\pi c}{l} \sin \frac{2n\pi t}{\tau} \qquad \dots \qquad \dots \quad (5)$$

which evidently expresses a motion of the simple two-step zig-zag type, the ratio v_a/v_b being equal to the ratio in which the point $x = c$ divides the *half*-length of the string. The motion at other points may be found by graphical methods. If c be nearly equal to $l/2$, the amplitude of the second, fourth harmonics etc., becomes large in comparison with the amplitude of the other partials, and may even transcend that of the fundamental component of the vibration. This is clear from (4) above.

Case of three equal discontinuities :

In this case, the cosine components vanish only if at the instant at which two of the discontinuities pass each other, the third discontinuity is at the end of the string. This instant is taken as the origin of time. If the two discontinuities pass each other at the point $x = l/3 \pm 2b$ (where $b \lessgtr l/6$), it may be readily shown as in the preceding case that the motion at the point $x = l/3 \mp b$ is of the simple two-step zig-zag type, the ratio v_a/v_b being equal to the ratio in which that point divides the *third* of the length of the string.

If b be sufficiently small, the amplitudes of the third, sixth harmonics etc., become large in comparison with that of the fundamental and other components.

Cases of four, five or more equal discontinuities.

As regards these cases, it must suffice to mention the following kinematical result which emerges from a detailed discussion. If r, the number of discontinuities be a *prime* integer, e.g., 2, 3, 5, 7, 11 etc., it is always possible by suitably choosing the initial position of the discontinuities to secure that the motion at any specified point on the string (not lying near one end) is of the simple two-step zig-zag type. If the point specified lies near one of the points of section of the string into r equal parts, the rth, $2r$th, $3r$th harmonics etc., have relatively large amplitudes in the type of vibration thus set up.

But the case is entirely different when r, the number of dis-continuities is not a prime integer, e.g., 4, 6, 8, 9 or 10, and is therefore a multiple of some smaller number. It is then found that no disposition of the discontinuities can secure a simple two-step zig-zag type of motion at any point lying elsewhere than within certain limited sections of the string. A simple example will make this clear. With 6 discontinuities on the velocity-diagram, it is kinematically possible to secure a two-step zig-zag type of motion at the bowed point if it lies in the vicinity of the node $l/_6$ on either side and the 6th harmonic is then powerful, in the motion elicited. If it lies elsewhere, that is, in the vicinity of the nodes $l/_3$ or $l/_2$, the motion is necessarily of a more complicated type, a four-step zig-zag and a six-step zig-zag being respectively the types of vibra-tion of minimum complexity admissible at the bowed point in the two cases.

4. *Vibrations with Missing Partials.*

We may now easily pass to the cases in which owing to the coincidence of the bowed point with a point of rational division of the string, certain partials fail to be maintained. It is obvious that the falling out of these partials leaves the motion at the bowed point unaffected. The mode of vibration of the string in these cases, can be very simply derived from the corresponding 'irrational' types by simple subtraction of the partials having a node at the bowed point. It will be shown here how this may be done in the most general case of discontinuous vibration. Assume that the bowed point coincides with a node of the sth harmonic which of course is also a node of the $2sth$, $3sth$ harmonics etc. Taking the analysis of the velocity diagram given in (1) we have·

$$A_1 = -\frac{2}{\pi}\left[d_1 \cos\frac{\pi C_1}{l} + d_2 \cos\frac{\pi C_2}{l} + \text{etc} \right]$$

$$A_s = -\frac{2}{s\pi}\left[d_1 \cos\frac{s\pi C_1}{l} + d_2 \cos\frac{s\pi C_2}{l} + \text{etc} \right]$$

$$= -\frac{2}{\pi}\left[\frac{d_1}{s}\cos\frac{\pi C_1}{l/s} + \frac{d_2}{s} \cos\frac{\pi C_2}{l} + \text{etc} \right] \qquad \dots \qquad \dots \text{(6)}$$

Similarly we have

$$A_n = -\frac{2}{n\pi}\left[d_1 \cos \frac{n\pi C_1}{l} + d_2 \cos \frac{n\pi C_2}{l} + \text{etc} \right]$$

$$A_{ns} = -\frac{2}{n\pi}\left[\frac{d_1}{s} \cos \frac{n\pi C_1}{l/s} + \frac{+d_2}{s} \cos \frac{n\pi C_2}{l/s} + \text{etc} \right] \quad \dots \quad (7)$$

Now the summation of the series $\sum\limits_{n=1}^{n=\infty} A_n \sin \frac{n\pi e}{l}$ of which A_1 $\sin \frac{\pi e}{l}$ is the leading term gives us the original velocity diagram. From (6) and (7), it is evident that the subordinate series $\sum\limits_{n=1} A_{ns} \sin \frac{n\pi e}{l/s}$ of which $A_s \sin \frac{s\pi e}{l}$ is the leading term is of analogous type and gives us a velocity diagram with discontinuities d_1/s, d_2/s, etc., the ordinates of which have to be subtracted from the original diagram. The lines in the principal and subordinate diagrams are obviously inclined to the x—axis at the same angle, and the resulting figure in which the missing partials are excluded is therefore made up of straight lines parallel to the x—axis and separated by discontinuities. From this diagram, the nature of the vibration at any point of the string may be readily found by graphical methods.

It is evident that the effect of the falling out of the partials having a node at the bowed point is to introduce into the velocity diagram of the string, a number of discontinuities which are smaller than the original discontinuities in the ratio $1/s$ and of opposite

5. Vibrations with the Missing Partials Partially restored.

A very complete graphical treatment of these cases is given in the monograph where they are referred to as 'transitional modes of vibration.' They are intermediate in form between the various types of vibration discussed in the third and fourth sections of this paper, and have the distinguishing characteristic that the speed of the bowed point in the forward motion is generally constant, but that in the back-ward motion is not uniform. Generally speaking, they are of an unsymmetrical type, that is involve both sine and cosine functions

of the time. From a musical point of view, they are of great
importance, and a fuller discussion of their features is being published
elsewhere. Here, it must suffice to remark that the general analytical
formula for discontinuous vibration given in Section 2 is sufficient
to cover these cases as well.

MiTRA.
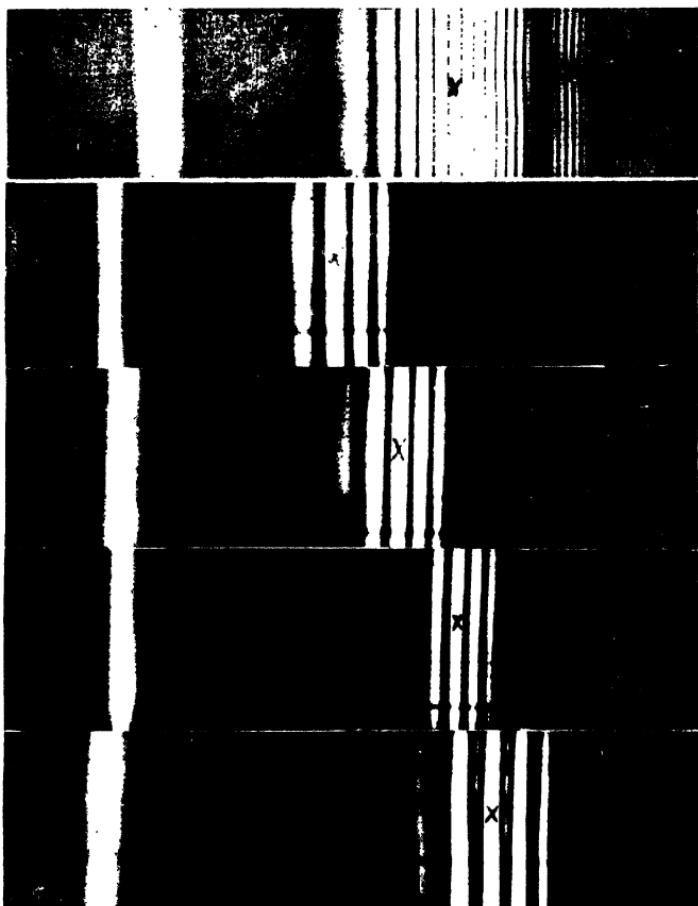
Illustrating the asymmetry of the fringes and of the illumination curves
in oblique diffraction by a reflecting surface consisting of two parts
(Fig. I to IV) or three parts (Fig. V.) in the same plane.

On the Asymmetry of the Illumination-curves in Oblique Diffraction.[*]

BY

Sisirkumar Mitra, M.Sc.

Introduction.

In the Phil. Mag. for May 1911, C. V. Raman has given the results of a photometric study of the unsymmetrical diffraction-bands due to an obliquely held rectangular reflecting surface previously observed by him.[†] The measurements showed a very marked asymmetry in the distribution of intensity in the diffraction pattern, the theoretical explanation of which is discussed in the papers quoted. The following were the principal conclusions arrived at by Raman as the result of the quantitative experimental study of the case :—

(*a*) The illumination at the points of minimum intensity in the diffraction pattern is zero at all angles of incidence, and the positions of the minima are accurately given by the formula

$$\delta = \pm \pi. \; \pm 2\pi, \; \pm 3\pi, \; \&c..$$

where $\delta = \dfrac{\pi a}{\lambda}$ (sin i — sin θ), a being the width of the aperture, λ the wave-length, and i, θ the angles of incidence and diffraction respectively ; the fringes are wider on the side on which $\theta > i$ and their number is limited on that side, as θ cannot be greater than $\dfrac{\pi}{2}$.

(*b*) The formulæ of the usual type ($I = \sin^2 \delta/\delta^2$) for the illumination in the pattern fails to represent the observed intensity-curves at oblique incidences except in regard to the position of the minima ($\delta = \pm \pi, \pm 2\pi, \&c.$). The intensities at corresponding points on either side of the central fringe for which the values of δ are numerically the same are *not* equal.

[*] Reprinted from the Philosophical Magazine for January, 1918.

[†] C. V. Raman, M.A., "On the Unsymmetrical Diffraction Bands due to a Rectangular Aperture," Phil. Mag. Nov. 1906. See also Phil. Mag. Jan. 1909

(c) The observed distribution of intensity was found to fit in with the theoretical formula, if the latter is multiplied by a factor proportional to the square of the cosine of the obliquity, which, of course, is not the same at all points in the diffraction pattern. In other words, the ordinates of the illumination curve were found to be proportional to the expression $\cos^2\theta \sin^2 \delta/\delta^2$.

The question arises whether these results, practically those indicated in (b) and (c) above, are peculiar to the case of a surface of rectangular form, or whether similar phenomena might be expected with other forms of surface as well. The cases which it seemed of particular interest to examine are those in which the reflecting surface is not a single individual area but consists of two, three, or more parallel elements lying in the same plane. A satisfactory surface of this kind which can be used at very oblique incidences may be prepared by etching out deep grooves on the optically plane surface of a thick plate of glass with hydrofluoric acid, the edges of the reflecting strips left on the surface being subsequently ground so as to be sharp, straight, and parallel. I have prepared several such surfaces containing two and three equidistant reflecting strips respectively. By placing one of these on the table of a spectrometer, the diffraction pattern produced by reflexion at a very oblique incidences may be readily observed through the telescope of the instrument. The present paper describes the results of the quantitative study of the phenomena thus obtained. Incidentally the opportunity has also been taken of testing the results obtained by Raman for the case of a single aperture using improved optical and photographic appliances. The experiments and determinations have throughout been made using monochromatic light. This was secured by illuminating the slit of the spectrometer with light of a definite wave-length isolated by a monochromator from sunlight or arc light. ·

Unsymmetrical Interference-fringes due to two parallel apertures.

Fig. 1. (Pl.) reproduces a photograph of the diffraction pattern due to a surface containing two reflecting elements each of width 0·48 cm., and 3·60 cm. apart. The direct image of the slit of the spectrometer also appears in the figures to the left of the diffraction-pattern The photograph is reproduced from a dense negative taken to show the perfect blackness of the minima of illumination, and the progressive increase (from right to left) in the width

of the interference-fringes of the light diffracted by the two reflecting elements. It was obtained by replacing the telescope of the spectrometer by a camera with a lens of long focus (176cm.). Figs. II, III, and IV reproduce three photographs taken at three different angles of incidence, the reflecting strips in this case being 0·754 cm. wide and 1·446 cm. apart. In all the figures, the central fringe of the pattern is indicated by a small cross ×. The asymmetry of the luminosity curve will be evident on comparing the brightness of the corresponding bands on either side of the central fringe; for instance, the second band on the right and the second band on the left in figs. III and IV, or the first band on the right and the first band on the left in fig. II.

The positions of the interference minima in the pattern are given by the formulæ of the usual type

$$\delta = \pm \frac{\pi}{3}, \quad \pm \frac{3\pi}{.}, \quad + \frac{.\pi}{.}, \text{ &c.}$$

where $\delta = \pi(a+b)(\sin i - \sin \theta)/\lambda$, a being the width of each of the apertures, b their distance apart, and i, θ, λ having their usual signifiance. To test whether the formula holds good at the oblique incidences used, the negatives were measured under a travelling microscope. In photograph I, the distances between the successive interference minima were determined to find whether the relations

$$\sin \theta_1 - \sin \theta_2 = \sin \theta_2 - \sin \theta_3 = \sin \theta_3 - \sin \theta_4, \text{ &c.,}$$

indicated by the formulæ were valid. The results are shown in Table I.

TABLE I.

$a = 0·48$ cm. $b = 3·60$ cm.

Minima on the right of the central fringe.	Observed value of $\sin \theta_{n+1} - \sin \theta_n$ × constant.	Minima on the left of the central fringe.	Observed value o $\sin \theta_n - \sin \theta_{n+1}$ × constant.
$\sin \theta_6 - \sin \theta_5$	0·712	$\sin \theta_1 - \sin \theta_2$	0·705
$\sin \theta_4 - \sin \theta_3$	0·703	$\sin \theta_2 - \sin \theta_3$	0·710
$\sin \theta_3 - \sin \theta_2$	0·711	$\sin \theta_3 - \sin \theta_4$	0·711
$\sin \theta_2 - \sin \theta_1$	0·711	$\sin \theta_4 - \sin \theta_5$	0·710
...	...	$\sin \theta_5 - \sin \theta_6$	0·713

For photographs II, III, and IV, the actual values of θ for the interference minima were calculated from the known constants a, b, i, λ and compared with the observed values. These are shown in Tables II, III, and IV.

TABLE II.

$a = 0.754$ cm. $b = 1.446$ cm. $\lambda = .0000435$ cm. $i = 89° 15'.27$.

Interference Minima.	Calculated diffraction angle, $90° - \theta$ (in minutes).	Observed diffraction angle, $90° - \theta$ (in minutes).
2nd on the left.	35'.98	35.'85
1st „ „ ,	41'.97	41.'83
1st „ „ right.	47'.21	46.'95
2nd „ „ „	.51'.92	52.'05
5th „ „ „	64'.02	63.'95

TABLE III.

$a = 0.754$ cm. $b = 1.446$ c.m. $\lambda = .0000435$ cm. $i = 89° 22'.75$.

Interference Minima.	Calculated $90° - \theta$.	Observed $90° - \theta$.
2nd on the left	26'.16	26'.12
1st „ „ „	33'.88	33'.85
1st „ „ right.	40'.25	40'.09
2nd „ „ „	45'.69	45'.70
5th „ „ „	58'.90	58'.75

TABLE IV.

$a = .754$ cm. $b = 1.446$ cm. $\lambda = .0000453$ cm. $i = 89° 29'.90$.

Interference Minima.	Calculated $90° - \theta$.	Observed $90° - \theta$.
1st on the left.	25'.72	25'.62
1st on the right.	33'.75	33'.45
2nd „ „ „	40'.08	39'.94
5th „ „ „	54'.94	54'.96

(a)

(b)

(c)

Illustrating Oblique Diffraction by a Semi infinite
Screen.

The Asymmetry of the Illumination-Curves.

As remaked above, there is a very marked difference in the luminosity of the corresponding bands on either side of the central fringe of the pattern due to the reflecting surface of two elements. Similar effects are also noticeable when the reflecting surface consists of three elements. Fig. V in the Plate reproduce two photographs obtained with a reflecting surface consisting of three elements. The difference between the intensities of the 2nd principal maximum on either side of the central one is very evident in the reproductions and might be made out even in respect of the secondary maxima on either side. This asymmetry demands an explanation. As is shown by the measurements given in Tables I. to IV., the positions of the minima of illumination are in good agreement with these calculated from the formula of the usual type, which are obtained on the assumption that each of the elements into which the reflecting surface may be divided diffracts light strictly in proportion to its area, and that the phase and intensity of the disturbance incident on the surface are the same as when the waves travel undisturbed. Further, the intensities at the points of minimum illumination are shown by observation and by the photographs to be zero, in agreement with the results indicated by these formulæ. On the other hand, the difference in the intensity at corresponding points of the pattern on either side of the central fringe remains unexplained according to such formulæ unless regarded as an *obliquity effect*.

A series of comparisons of the intensities of corresponding bands on the two sides of the pattern has been made for the cases in which the reflecting surface consists of one, two, and three reflecting elements respectively, for various angles of incidence. For this purpose, I have used a rotatingsector photometer of the Abney type supplied by Messrs. Adam Hilger, in which the free disk, which can be adjusted by handle while in rotation, is smaller in radius than the fixed disk. The sectors when in rotation thus present two annuli of different intensities, the ratio of which can be ajdusted at pleasure by moving the handle of the instrument. The disk of the photometer is placed at the focal plane of the observing telescope, so that the diffraction pattern can be seen through it with an eyepiece, the fringes on the brighter side being observed through the inner annulus of the disk, and those on the

fainter side through the outer annulus. To enable the intensities
at corresponding points on the two sides of the pattern to be com-
pared, a screen with two vertical slits is interposed immediately in
front of the photometric disk so as to cut off everything except
the regions under observation, which are then adjusted to equality
of brightness by moving the handle of the photometer. Several
readings can be taken in succession and their average struck. The
diffraction angles θ and θ' of the two bands under comparison may
then be measured under a micrometer eyepiece. Tables V., VI.,
VII. show the observed ratios of the illumination and those calcula-
ted on the assumption that the formulæ for illumination includes
a factor proportional to the square of the cosine of the obliquity.
It is seen that the agreement is good except when the ratio is so
large that it connot be measured accurately, owing to the near
approach of the fainter band towards the direct image of the slit.

Table V.

Single Reflecting Surface, width 0·90 cm.

Ratio of intensity of the first band on the right and the first band
on the left.

Angle of Incidence.	Observed ratio of illumination.	Calculated ratio $\cos^2\theta \cdot \cos^2\theta'$.
88°42′	1·80	1·78
88°53′	2·31	2·46
88°56′	2·81	2·89
88°4′	4·09	4·21

Table VI.

Reflexion granting of two elements.

$a = 0.754$ cm. $b = 1.446$ cm.

Angle of Incidence.	Ratio of the 1st maxima on the right and left.		Ratio of the 2nd maxima on the right and left.	
	Observed.	Calculated.	Observed.	Calculated.
89°6′	1·43	1·40	2·09	2·01
89°23′	2·25	2·01	3·91	4·88
89°28′	2·37	2·44	7·50	11·06

TABLE VII.

Reflexion granting of three elements.

$a = 0.440$ cm. $b = 0.741$ cm.

Angle of Incidence.	Ratio of the 1st two Principal Maxima on either side of the central one.		Ratio of the 2nd two Principal Maxima on either side of the central one.	
	Observed.	Calculated.	Observed	Calculated.
88°49′	1·48	1·43	1·83	1·70
89°5′	1·63	1·71	2·90	3·15
89°27′	2·65	2·87	×	more than 100 times.

Summary and Conclusion.

1. The unsymmetrical interference fringes of the light obliquely diffracted by two parallel reflecting surfaces in the same plane have been observed and photographed.

2. The illumination curve in the diffraction pattern (of the Fraunhofer class) due to an obliquely-held reflecting surface (which may consist of two or more separate parts in the same plane), is found to be markedly asymmetrical, corresponding points on either side of the central fringe being of very different intensities. As the positions of the points of minimum (*i.e.* zero) illumination are found to be in close agreement with those given by the formula of the usual type, the asymmetry of the illumination curve may be explained as due to the varying obliquity at different points in the diffraction pattern. Measurements of the ratio of the intensities at corresponding points have been made with reflecting surfaces of rectangular form or consisting of either two or three elements in the same plane for various angles of incidence; the results show that the expression for the illumination at any point of the diffraction pattern contains a factor proportional to the square of the cosine of the obliquity at such point.

The experiments and observations described in the note were carried out in the Palit Laboratory of Physics. The writer hopes to carry out further work on the subject of oblique diffraction by various forms of aperture, and particularly in regard to the positions of the points of maximum intensity in the pattern, which would no doubt differ from those given by the usual formulæ owing to the asymmetry of the illumination curves.

On Sommerfeld's Treatment of the Problem of Diffraction by a Semi-Infinite Screen.[1]

BY

SISIRKUMAR MITRA, M.Sc.

§1. Introduction.

The rigorous treatment of the effect of obstacles on the propagation of light, considered as a boundary value problem in analysis, has received much attention from mathematical physicists during recent years.[2] The particular case of the diffraction of light by a semi-infinite perfectly reflecting screen for which the complete solution was first given by Sommerfeld[3] (and later also by Carslaw[4]) has been more recently dealt with by Lamb[5] in a paper characterised by very simple and elegant mathematical analysis. As remarked by Lamb in his paper, the principal interest to the physicist of investigations as these, lies in the fact that they afford a check on the accuracy of the results obtained by less rigorous mathematical methods, and also enable a comparison of the theory with experiment to be carried out for cases in which the ordinary treatment can hardly be regarded as applicable. A comparison of the results of the approximate theory with those deduced from the rigorous analysis for the case of the semi-infinite screen has been made by Sommerfeld himself, and also by Drude[6] who has used the Cornu spirals with good effect, in his discussion of the value of Sommerfeld's integrals. According to these writers, Kirchhoff's formula should give the value of the intensity of illumination with sufficient accuracy when the angle of diffraction is small, that is, at all points of the field (except very near the edge of

[1] Reprinted from the Philosophical Magazine for January, 1919.

[2] An excellent summary of the literature on the subject with references to the original papers will be found in the article by Epstein in the Encyclo. Math. Wiss (Section on Wave Optics, 1914).

[3] Sommerfeld, Math. Annalen, t. XLVII, page 317 (1895).

[4] Carslaw, Proc. Lond. Math. Soc., (2) t XXX, page 121 (1899).

[5] Lamb, Proc. Math. Soc., (2) t. IV page 190 (1906).

[6] Drude, Theory of Optics, English translation by Mann and Millikan (1902) page 203.

the screen) which are not far removed from the boundary between
light and shadow; but for large angles of diffraction, Kirchhoff's
formula is inapplicable. It appears, however, from a careful examina-
tion of the formulæ given by Sommerfeld and Drude that the state-
ment made by them on this point requires to be qualified in one
important respect. I propose in the present paper to show by a
detailed discussion that *when the screen is held very obliquely in the
path of the incident waves*, the rigorous treatment gives results differ-
ing from those of the approximate theory *even in regard to small
angles of diffraction*. Experimental work recently carried out by me
and described in the course of the paper confirms this, and shows that
the approximate theory of diffraction fails to represent the facts
correctly under these conditions. Incidentally it is found, that in the
case of light polarised in a plane parpendicular to that of incidence,
the boundary condition at the screen assumed by Sommerfeld leads
to results differing very widely from the observed optical behaviour
of any actual screen at very oblique incidences, and a suggestion is
made as to the manner in which the rigorous solution should be
modified in order to secure an agreement with the results obtained
experimentally in this case.

§2. *Theory.*

It is convenient here to state Sommerfeld's results in the
simplified form obtained by him from a semiconvergent expansion
of the integrals representing the complete solution.

This is

$$S = \cos\left[\frac{2\pi}{\lambda} r \cos(\phi - \phi') + nt\right] \dotplus \cos\left[\frac{2\pi}{\lambda} r \cos(\phi + \phi') + nt\right]$$

$$+ \frac{1}{4\pi} \sqrt{\frac{\lambda}{r}} \left[\dotplus \frac{1}{\cos\frac{\phi + \phi'}{2}} - \frac{1}{\cos\frac{\phi - \phi'}{2}}\right] \cos\left[\frac{2\pi}{\lambda} r + \frac{\pi}{4} - nt\right]$$

whiere S is the light disturbance, ϕ' and ϕ are respectively the
angles made by the incident rays and by the radius vector with the
plane of the screen, and r is the distance of the point of observation
from the edge of the screen. The alternative signs refer to the
state of polarisation of the incident light. The upper sign should be
taken in the case when the incident light is polarised in a plane
perpendicular to the edge of the screen, *i.e.*, when the electric vector

is parallel to the edge, and the lower sign in the case when the light is polarised in a plane parallel to the edge, *i.e.*, when the magnetic vector is parallel to the edge. The first and second terms in the expression represent the incident and the reflected waves respectively, while the third term gives the wave of diffraction. In the region of shadow, only the third term should be taken into account; in the region of transmission we have to take the first and third terms only, while in the region of reflexion all the three terms in the expression for the light disturbance have to be retained. Thus beyond the path of the rays determined by geometrical optics, there is a wave of diffraction whose phase is determined by the factor

$$\cos \left[\frac{2\pi}{\lambda} r + \frac{\pi}{4} - nt \right]$$

and whose amplitude by the factor

$$\frac{1}{4\pi} \sqrt{\frac{\lambda}{r}} \left[\pm \frac{1}{\cos \frac{\phi + \phi'}{2}} - \frac{1}{\cos \frac{\phi - \phi'}{2}} \right] \cos \left[\frac{2\pi}{\lambda} r + \frac{\pi}{4} - nt \right]$$

The lines of equal phase are circles round the point $r=0$, so that from this point rays start out in all sides in the direction of the radius vector straight on as if the edge of the screen were a linear source of light. The intensity of these cylindrical waves is however not same in all directions. It is greatest near the region of the two boundaries separating the different parts of the field and gradually diminishes as we go away from these boundaries.

As is well known, the most remarkable result indicated by Sommerfeld's analysis and which is in substantial agreement with the experimental observations of Gouy, Wien and others, is that the amplitude of the diffracted waves is different for light polarised in and at right angles to the plane of incidence. This is sufficiently clear from the expression given above. For small angles of diffraction however, that is in the neighbourhood of the planes defined by $\phi = \pi - \phi'$ and $\phi = \pi + \phi'$, the difference in the magnitudes of the two components is generally, quite negligible. For, in the neighbourhood of the first plane, $\dfrac{1}{\cos \frac{\phi + \phi'}{2}}$ in numerically very large compared with $\dfrac{1}{\cos \frac{\phi - \phi'}{2}}$, and similarly in the neighbourhood of the second

plane $\dfrac{1}{\cos\frac{\phi-\phi'}{2}}$ is very large compared with $\dfrac{1}{\cos\frac{\phi+\phi'}{2}}$. Accord-

ingly, in the neighbourhood of these two planes, it is ordinarily sufficient to retain one of the two terms and neglect the other in the expression for the amplitude of the diffracted waves, which is thus numerically the same for both states of polarisation, and for the region considered is in substantial agreement with that found from the approximate theory. But the preceeding argument fails entirely when ϕ' is nearly equal either to π or zero, that is when the incidence of the light at the secreen is very oblique. For the two terms

$\dfrac{1}{\cos\frac{\phi+\phi'}{2}}$ and $\dfrac{1}{\cos\frac{\phi-\phi'}{2}}$, are then always of comparable magnitude,

and have both to be retained. It is thus clear that some special features are to be expected when the incidence of light on the screen is very oblique.

We may now consider separately the two cases in which ϕ' is nearly equal to π and zero respectively, as they present distinctive features. When ϕ' is nearly equal to π (see Fig. 1a and 1b), by far the largest part of the field is occupied by the region of trans- mission (marked II in Fig.), and the remaining part of the field is equally divided between the regions of shadow and of reflexion (marked I and III respectively in the Fig.) which lie on opposite sides of the screen and are completely separated by it. If we wish to observe the phenomena in the neighbourhood of the boundaries separating the different parts of the field, we have two distinct choices open to us.

We may either study the phenomena near the boundary separating regions I and II where ϕ is nearly equal to 2π (Fig. 1a), or near the correspoding boundary on the other side of the screen between regions II and III in which case ϕ is nearly equal to zero (Fig. 1 b). The case in which ϕ is nearly equal to 2π is the simpler of the two, as we are then concerned only with the transmitted and diffracted wave trains. Putting $\phi'=\pi-\alpha$ and $\phi=2\pi-\beta$ where α and β are small

angles (fig 1a) it is found that $\dfrac{1}{\cos\frac{\phi+\phi'}{2}}$ and $\dfrac{1}{\cos\frac{\phi-\phi'}{2}}$ are of

comparable magnitude. Sommerfeld's formula thus leads to the striking result that in the case of a very obliquely held screen, the intensity of the diffraction fringes seen near the boundary between light and shadow should depend to a very considerable extend upon the plane of polarisation of the incident light and should be quite different from that given by Kirchhoff's formula.

Fig. 1a.
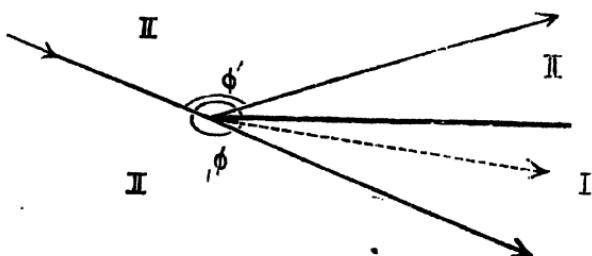
Fig. 1b.
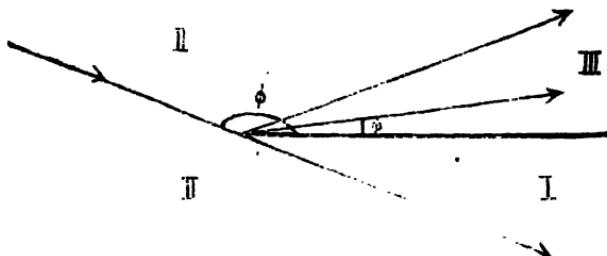
Similarly on the other side the screen (Fig. 1b) putting $\phi'=\pi-\alpha$ and $\phi=\beta$ where α and β are small angles, we may work out the expression for the light disturbance. If $\beta>\alpha$ we are only concerned with the interference of the incident and diffracted wave trains, while if $\beta<\alpha$ we have to consider the stationary waves formed by the interference of incident and reflected wave trains as modified by the superposition of the cylindrical waves radiated by the edge of the screen. If the light be polarised in the plane of incidence, we find that when ϕ is vanishingly small, the expression for the light disturbance in zero. This shows that the surface of the mirror is a nodal plane for the light vector in this case. On the other hand if the

light be polarised prependicular to the plane of incidence, it is found
that for vanishingly small value of ϕ, that is, along the surface of
the mirror, the intensity of the diffracted rays does *not* vanish and
that, moreover, the incident and reflected wave trains re-inforce one
another, the light vector being thus a maximum at the surface of the
mirror. The last-mentioned result is a direct consequence of the
boundary condition $\left(\dfrac{\partial s}{\partial z}=0\right)$ at the surface of the mirror assumed
by Sommmerfeld as the basis of his work, but is contrary to the
observed optical behaviour of any actual screen at very oblique in-
cidences. In practice any polished surface is at such incidences,
nearly a perfect reflector, but, as shown hy Lloyd's experiment, both
the electric and magnetic vectors in the incident and reflected waves
are of opposite signs at the surface irrespective of the plane of polari-
sation of the incident light, and their resultant is zero.

The preceding discussion makes it clear that the solution obtained
with the boundary condition $\dfrac{\partial s}{\partial z}=0$ for the light vector at the surface
of the mirror is entirely inapplicable under experimental conditions
for oblique incidences. The solution obtained with the boundary
condition $(s=o)$ may however for our present purpose be regarded as
practically valid at all incidences for any screen which is a sufficiently
near approach to a perfect reflector, provided *the light bepolarised in the
plane of incidence*. The experimental investigation described in
the present paper shows that *the same solution may also be re-
garded as applicable for light polarised at right angles to the plane
of incidence* provided the incidence be, very oblique, and attention
is confined to the phenomena observed at small angles of diffraction.

§3. *Experimental Method and Results.*

The diffraction fringes of the Fresnel type bordering the
shadow of an obliquely held screen may be observed with the
arrangement shown diagramatically in Fig. 1a. A front-silvered
glass plate bordered by parallel straight edges may be used
as the screen. The first edge diffracts the incident light, and by
observing the fringes near the surface of the mirror at the second

edge, the necessity of using an infinitely extended screen is avoided. The contrast between the maxima and minima of illumination is found to be not entirely independent of the inclination of the surface of the screen to the direction of the incident rays. When the inclination is considerable, the fringes at the edge of the shadow are of course, of the usual Fresnel type, few in number and very diffuse. But as the surface is gradually brought up to the position in which it just begins to graze the incident light, the contrast between the maxima and minima of illumination in the fringes gradually increases, their relative positions remaining unaltered and at the same time the falling off of the intensity to zero inside the geometrical shadow of the screen becomes more rapid than in the diffraction fringes of the ordinary type due to a normally held screen. On examination of the fringes through a nicol, it is found that the intensity of the fringes is independent of the plane of polarisation of the incident light.

A detailed comparison has been carried out between the position and the intensity of the fringes as observed experimentally with those calculated from the theoretical expression

$$S = \cos\left[\frac{2\pi}{\lambda} r \cos(\phi - \phi') + nt\right]$$

$$+ \; 4\pi \; \sqrt{\frac{\lambda}{r}} \left[\frac{1}{\cos\frac{\phi+\phi'}{2}} - \frac{1}{\cos\frac{\phi-\phi'}{2}}\right]$$

$$\cos\left[\frac{2\pi}{\lambda} r + \frac{\pi}{4} - nt\right]$$

which satisfies the boundary condition $s = o$ at the surface of the mirror. When ϕ' in much less than π, the intensity curve given by the expression is practically the same as that obtained from usual Fresnel integrals and is shown in the dotted line in Fig. 2 (c).* As ϕ' is gradually increased so as to approach

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* The asymptotic expansion given by Sommerfeld is inapplicable over a very small part of the field on either side of the boundaries $\phi = \pi + \phi'$ and $\pi - \phi'$. A small part of each of the curves shown in the Figure was accordingly filled in freehand so as to represent as closely as possible the general trend of the curve.

the value π, the maxima and minima of illumination remain unaltered in position, but the contrast between them gradually increases. The full line in Fig. 2 (c) shows the calculated intensity curve in the limiting case in which ϕ' is equal to π and the screen just grazes the incident light. The illumination is seen to be zero on the surface of the mirror.

Fig. 2.

the value π
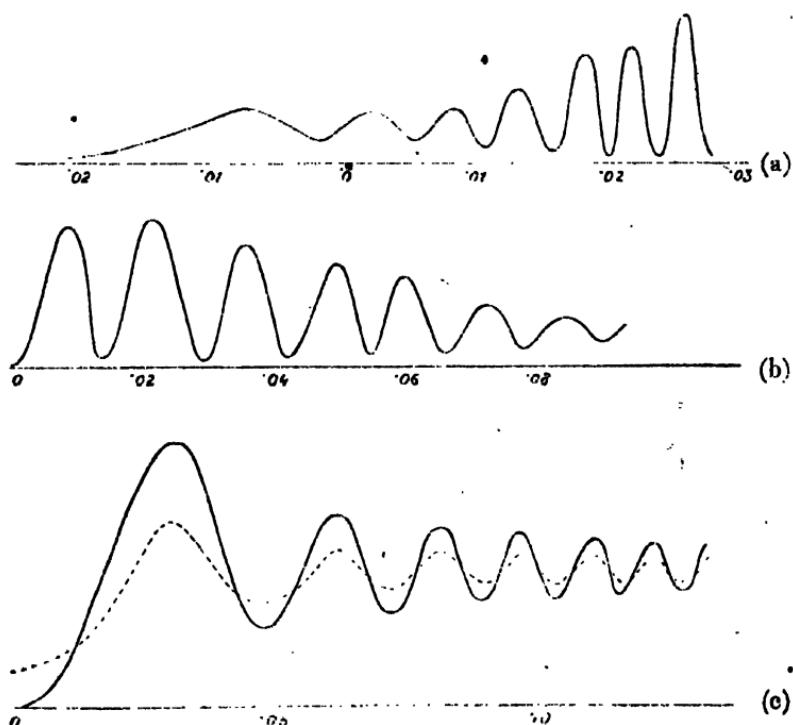
Table I shows in the first column the intensities of the maxima and minima in the diffraction fringes of the Fresnel type due to a normally-held screen and in the second column those due to a screen grazed by incident rays, the intensity in the incident waves being taken as unity. The positions of the minima are also given, which are of course the same in both cases.

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TABLE I.

$\lambda=4377 \times 10^{-8}$ cm, $r=30.75$ cm, $\phi'=179^\circ\ 51'\ 56$.

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Calculated intensity at the maxima and minima Case I.		Calculated intensity at the maxima and minima case II.		Calculated distances of the minima from the edge case I and II. $\times \sqrt{\dfrac{2}{r\lambda}}$ in cm	Observed distances of the minima from the edge in case II. $\times \sqrt{\dfrac{2}{r\lambda}}$ in cm
Max.	Min.	Max.	Min.		
1·37		1·95			
	·78		·58	1·871	1·853
1·20		1·41			
	·84		·69	2·739	2·714
1·15		1·30			
	·87		·76	3·391	3·340
1·12		1·28			
	·89		·79	3·937	3·930
1·11		1·22			
	·90		·80	4·416	4·393
1·10		1·19			
	·91		·81	4·848	4·830
1·09		1·18			
	·92		·82	5·244	5·216

The fourth column in the Table gives the positions of the maxima and minima of illumination measured on a photograph taken of the fringes due to a screen grazed by the incident rays, and these agree closely with the theoretical values given in the third column.

The ratios of the intensity of illumination at the maxima and minima have been determined photometrically for comparison with the theoretical value shown in the first column of Table I. As mentioned above, the intensities were found to be independent of the plane of polarisation of the incident light. The method adopted for the photometric work was as follows. The incident light was plane polarised by passage through a nicol. Two narrow slits were mounted one above the other in the plane of the diffraction fringes, and a thin plate of mica of proper thickness (·032mm) was fixed up on the upper

one so oriented that it circularly polarised the light falling on it. The field was observed through an eyepiece and an analysing nicol mounted in a graduated circle. The lower and the upper slits were then respectively set on the 1st maximum and the 1st minimum, or the 2nd maximum and the 2nd minimum and so on. The illuminations of the upper and lower slits were equalised by rotating the analysing nicol, and the position of the analyser at which the light from the lower slit was extinguished was also noted. The angular difference of the two positions suffices to give the ratio of the illuminations. Thus if θ_1 and θ_2 be the two positions, and I_1 and I_2 the intensities of illumination of the upper and lower slit respectively,

$$\frac{I}{I_0} = \sin^2 (\theta_1 - \theta_2)$$

A correction was made for the loss of light in transmission through the mica sheet covering the upper slit.

TABLE II.

Ratio of the intensities of Minima and Maxima.	Observed value.	Value calculated from Sommerfeld's expression.	Value Calculated from Fresnel's Integrals.
1st min. and 1st max.　　...	·36	·30	·58
2nd min. and 2nd max.　　...	·53	·50	·67
3rd min. and 3rd max.　　...	·60	·57	·79

Table II shows the results of the photometric work which was carried out on the fringes obtained with monochromatic light. It was not found practicable to carry the measurement beyond the third fringe. The photograph reproduced in the Plate Fig. (a) clearly shows that the contrasts between the maxima and minima of illumination in this case are greater than in the diffraction fringes of the Fresnel type.

Two other cases besides that described above have been investigated experimentally. In one of these (shown diagramatically in Fig. 1 b) ϕ' was nearly equal to π and ϕ was small and positive.

The positions and magnitudes of the maxima and minima of illumination calculated from Sommerfeld's solution (the upper sign only being taken) and the experimental data are shown in Table III for comparison. The illumination-curve for this case is shown in Fig. 2 (*b*) and the photograph on which the measurements were made is reproduced in the Plate Fig. (*b*). In this case on the side of the screen under observation, only the reflected wave-front is limited by a boundary passing through the edge of the screen and suffers diffraction, and the general agreement between theory and experiment is only rendered possible by taking for the former, the rigorous solution obtained by Sommerfeld, the boundary condition $s=o$ being assumed to be the satisfied at the surface of the screen.

TABLE III.

$r=30.75$ cm, $\lambda=4380 \times 10^{-8}$ cm.

Intensity at the Maxima and Minima.		Calculated position of the minima. Distance from the edge in cm.	Observed position of the minima. Distance from the edge in cm.
Max.	Min.		
4·0	·18	0140	·0140
4·5	·02	0278	·0275
3·6	·20	0423	·0421
3·0	·21	0558	·0554
2·6	·32	·0682	·0681
1·8	·60	·0798	·0796
1·5	·64	·0912	·0907
1·3		·1014	·1008

The third case investigated is that shown diagramatically in Fig. 3. ϕ' is a small positive angle and ϕ is nearly equal to π.

In this case practically the whole of the field is divided equally

Fig. 3.
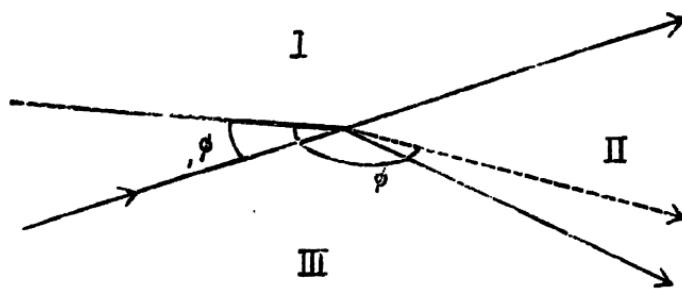
between the regions of shadow and reflexion (numbered I and III in the figures) and only a comparatively small portion (numbered II in the figure) is that occupied by the region of transmission. Both of the dividing boundaries fall in the region of the field under observation, and so far as this part of the field is concerned, the approximate theory of diffraction gives the same result as that found from the complete analytical solution satisfying the boundary condition $s=o$. The theoretical form of the illumination curve of a typical case of this kind is shown in Fig. 2 (a), and a photograph of the diffraction fringes is reproduced in the Plate Fig. (o). Table IV shows the calculated position of the minima of illumination and the experimental data for comparison. The agreement is satisfactory.

TABLE IV.

$\lambda = 4410 \times 10^{-8}$ cm, $\phi' = 19'\ 40.6''$, $r=3$ cm.

Calculated distances of the minima from the edge in cm.	Observed distances of the minima from the edge in cm.
—·0020	—·0020
·0051	·0051
·0103	·0104
·0149	·0152
·0193	·0194
·0232	·0235
·0269	·0273

§4. *Summary and Conclusion.*

The results arrived at from this investigation may be summarised
as follows :—

(1) When plane waves of light are diffracted by the edge
of a very obliquely held screen, the fringe system
observed *at and near the surface of the screen on either
side* shows features which require for their explanation,
the complete analytical investigation of diffraction
given by Sommerfeld. On one side of the screen, we
have the region of shadow, and adjoining it diffraction
fringes, the maxima and minima of illumination of
which show contrasts more marked than those in the
diffraction fringes of the Fresnel type, their positions,
however, being the same. This has been verified by
photometric observation. On the other side of the
screen, the fringes due to the interference of the direct
and reflected wave-trains are observed, and these
are modified by diffraction in a manner which can be
fully explained only in terms of the complete analytical
solution.

(2) The solution obtained by Sommerfeld with the boundary
condition $s=0$ at the surface of the mirror, agrees
with the results observed at oblique incidences, in the
part of the field under discussion, irrespective of the
plane of polarisation of the incident light.

The investigation described in this paper was carried out in the
Palit Laboratory of Physics.

BOTANY

COMMENTATIONES MYCOLOGICAE.

It is proposed to publish under the above general heading a series of papers, the results of investigations on the Fungus Flora of Bengal, carried out in the Biological Laboratory of the University College of Science, Calcutta. It is hoped that in this way a sufficient amount of material will be collected in the course of a number of years to make it possible to issue finally a connected account of the indigenous fungus flora, which, as may be imagined, is very rich, especially in those species which grow as parasites on phanerogams of nearly every description. The importance of such investigations in an agricultural country like Bengal cannot be exaggerated.

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P. BRÜHL,

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I.

MELIOLA GROWING ON *PHŒNIX SYLVESTRIS* AND *CITRUS MEDICA* (*VAR ACIDA*).

BY

S. N. BAL.

The particular Fungus described below was first noticed on the leaves of *Citrus medica* in the latter part September, 1918, and a little later the fruits of the same plant were also found to be attacked by the same fungus. Later in January, 1919, a fungus of precisely the same character was observed on the leaves of *Phœnix sylvestris*. The only difference observed was that the fungus growing on the leaves and fruits of Citrus medica became scarce with the advent of the cold weather, while that growing on the leaves of *Phœnix sylvestris* did not disappear at all and in fact gave good material to work with during the whole winter season.

External appearances :—Black sooty patches appearing on both sides of the leaves, more so on the upper side, and on the fruits of *Citrus medica*. No attack was observed on the fruits of *Phœnix sylvestris*. The mycelium is superficial and forms black spots on the surface of the leaves and fruits, and the main hyphae radiate

and send out short lateral branches which do not interweave, the whole thus forming a dendritic aggregate. The main hyphae are ·25 mm. to ·35 mm. long. They can easily be rubbed off the surface to which they are attached. It is then seen that the portions of the leaves attacked are yellow, due to the stoppage of the functions of chlorophyll. On examination of transverse sections passing through portions of leaves infested by the fungus it appears that the fungus growth is entirely superficial. It is therefore quite evident that the only harm the Fungus can cause is that which is done by its excluding light and air, and the chief damage due to the Fungus is obviously that caused by the spotting of the fruits.

The brownish black perithecia are spherical and their walls consist of a great number of cells convex outwards and with more or less hexagonal outlines. The perithecia are surrounded by unbranched, usually 4-6-cellular, straight appendages generally somewhat longer than the perithecia themselves. The asci are obovoid and 8-spored, the spores being 5-cellular. The asci can be seen only at the earliest stages of development; when the spores reach maturity no trace of the asci can be found, the walls of the asci evidently being absorbed when the spores reach maturity.

To see the asci clearly it is best to proceed as follows :—Small portions of leaves infested with the fungus are placed in a staining solution; several stains were used and a 25 per cent. alcoholic solution of safranin (1 gm. safranin in 100 cc. of 25 per cent. alcohol) gave the best results. The portions of leaves in the staining solution are kept overnight. Next day the mycelia bearing the perithecia are put on a clean slide, washed with alcohol, cleared with clove oil, again washed with absolute alcohol and xylol. A drop of canada balsam is placed on the material thus cleared and washed, and the perithecia are carefully pricked with a fine needle, put on a coverslip and examined. It is found that only the *young* spores take the stain, while the other portions remain unstained.

The young spores are almost colorless and only the septa dividing the spores into 5 cells and a very faint outer covering round the whole spores are visible under the microscope. The mature spores are brown. The spores are transversely septed and the sizes are as follows :—

Young spores   ...   4-5 μ long and 1·5—2 μ broad.
Mature spores   ...   6-7 μ long and 3-4 μ broad.

The sizes of the asci containing the very young spores are 7-8μ long and 5-6 μ broad.

Germination of spores :—The usual method of single spore germination was followed. A number of culture solutions of different nutritive media of different dilutions were used. All the nutritive media of different strengths containing complex organic substances such as cowdung, beef broth, beef-agar, peptone, decoction of *Citrus* leaves, and of leaves of *Phœnix sylvestris* etc. etc., did not give good results. The spores germinated most readily in culture media containing inorganic salts dissolved in distilled water, a solution containing KNO_3, Na_2HPO_4 and $(NH_4)_2SO_4$ in the proportion of 0·5 gm. each in 1000 c.c. of distilled water giving the best result. The germination of the spores was first observed to commence after 72 hours. After that the slides were examined every 24 hours. A clear idea of the rate of germination can be had by consulting the plate attached to this paper.

The Fungus is evidently identical or at least closely related to *Meliola amphitricha, Fries.*, which is described by Fischer in *Engler* and *Prautl's Pflanzenfamilien* as having 5-celled spores and straight upright appendages. It also reminds one strongly of the figure 218 c given on page 307 of the 1st *Abteilung* of the 1st *part* of *Engler* and *Prautl's Pflanzenfamilien.* The species there figured is *Meliola corallina,* Mont. *Meliola amphitricha* is said to occur on the leaves of various species growing in Cuba, Bonin islands and Tonkin, while *Meliola corallina* is stated to grow on *Drimys chilensis* in the island of Juan Fernandez. According to the figures cited the appendages of *Meliola corallina* are more or less curved.

Acknowledgments of thanks must be made to Dr. P. Brühl, University Professor of Botany, for his kind suggestions and criticisms.

My thanks are also due to Mr. Haraprasad Chaudhury, Sir Rashbehary Ghose Scholar in Botany, who assisted by me greatly in the laboratory part of the work.

EXPLANATION OF THE PLATE.

A. Leaf of *Citrus medica* (Var. acida), attacked by the Fungus. Natural size.

B. Leaf of *Phœnix sylvestris* attacked by the Fungus. Natural size.

C. The Fungus, × 20.

D. Perithecia with appendages, × 430.

E. 1. Young spore, × 430. 2. Mature spore, × 430.

F. Ascus containing 8 spores, × 430.

G. Spores in process of germination.

    1. Germinating as seen after   72 hours, spores × 430.

    2.   ...   ...   6 days   ...   × 430.

    3.   ...   ...   9 „   ...   × 430.

    4.⎫
    5.⎭   ...   ...   11 „   ...   × 430.

A

B

E

F

D

G

H